7. Quotients

Chef's table

This problem set focuses on quotients, but actually provides an excellent resource for you to review (almost) all the material we have so far seen in the course before we move on to an introduction to Algebraic Topology.

In many respects, you may consider this assignment as an *unofficial* take-home midterm and we encourage you to do so. For this reason we have decided to give you two weeks to work on these exercises, and we have also refrained from adding a challenge problem.

Some more specific comments: the first five exercises are rather straightforward, have short solutions and are intended as a warm-up. Among them, Problem 7.3 is close in spirit to Part Ib of the exam paper. Problems 7.6-7.7 are meant to help you acquire a working knowledge on quotients in some (fundamental!) examples, while 7.8 is a standard fact on the Alexandroff compactification (which will be presented during the next exercise class). Lastly, problems 7.9-7.10 collect some additional facts about quotients (which are important for you to keep in mind). Either of them is also close to Part IIa of the exam paper. Problem 7.10 is by far the most demanding exercise in this set, although it is supposed to be tractable by all students.

7.1. Important counterexample ✍. Consider the equivalence relation $\sim$ on $\mathbb{R}$, given by $x \sim y$ if and only if

$$x = y \qquad \text{or} \qquad |x| = |y| \text{ and } |x| > 1.$$

Let $Y := \mathbb{R}/\sim$ equipped with the induce topology. Show that Y is not a Hausdorff space.

7.2. Quotients and non-quotients ✍. Show that there is a quotient map $q\colon (-2, 2) \to [-1, 1]$, but not a quotient map $p\colon [-2, 2] \to (-1, 1)$.

7.3. Properties that descends to the quotient (or not) ✍. Let X, Y be topological spaces, and let $f\colon X \to Y$ be a continuous surjection. Assume that Y is equipped with the quotient topology, which means that a set $U \subseteq Y$ is open if and only if $f^{-1}(U)$ is open in X. Decide if the following statements are true or false: in case they are true, prove them; in case they are false, find a counterexample.

(i) If X is compact, so it is Y.

(ii) If X is Hausdorff, so it is Y.

(iii) If X is normal, then Y is Hausdorff.

(iv) If $|X| = \infty$, then $|Y| = \infty$.

(v) If X is connected, so it is Y.

(vi) If X is a metric space, so it is Y.

7.4. Connected components and quotients. Let X be a topological space, and assume that all connected components of X are open. Let $q\colon X \to Y$ be a quotient of X. Show that the connected components of Y are also open.

Note: It is not always true that all connected components are open, an example is $\mathbb{Q} \subseteq \mathbb{R}$ with the induced topology.

7.5. Saturated subsets. Let $q\colon X \to Y$ be a quotient. We say that a subset $A \subseteq X$ is *saturated* if $q^{-1}(q(A)) = A$. Show that if q is an open map and $A \subseteq X$ is a saturated subset, then also $\overline{A}$ and $\operatorname{int}(A)$ are saturated. Give an example where the map q is not open and the above is false (i.e. there is a saturated set B such that $\overline{B}$ or $\operatorname{int}(B)$ are not saturated).

7.6. The torus. Prove that the topological product space $X_1 = S^1 \times S^1$ is homeomorphic to the quotient $X_2 = Q/{\sim}$ obtained by considering on $Q = [0,1]^2$ the equivalence relation $(s,0) \sim (s,1)$ for all $s \in [0,1]$, $(0,t) \sim (1,t)$ for all $t \in [0,1]$. These are two possible (equivalent) definition of the torus T^2.

7.7. The real projective space. Prove that the topological spaces X_1, X_2 and X_3 defined below are homeomorphic. These are indeed three possible (equivalent) definitions of the projective space $\mathbb{P}^2(\mathbb{R})$ (see Lecture 12).

(i) Let $S^2 \subseteq \mathbb{R}^3$ be the two-dimensional unit sphere and consider on S^2 the relation $\sim$ that identifies the antipodal points on S^2, i.e. $u \sim v$ if and only if $u = v$ or $u = -v$. Then X_1 is defined as the quotient $X_1 := S^2/{\sim}$.

(ii) Consider the two-dimensional unit disk $D^2 \subseteq \mathbb{R}^2$ and the relation $\simeq$ on D^2 that identifies the antipodal points on its boundary, i.e. $u \simeq v$ if and only if $u = v$ or $u = -v$ with $u, v \in \partial D^2$. The topological space X_2 is then the quotient $X_2 := D^2/{\simeq}$.

(iii) Let $\mathcal{L}$ be the set of lines in $\mathbb{R}^3$ passing through the origin. Given $L_1, L_2 \in \mathcal{L}$, let $0 \leq \alpha \leq \pi/2$ be the angle between L_1 and L_2 and define $d(L_1, L_2) := \alpha$. Then we define the topological space X_3 as $\mathcal{L}$ with the topology induced by d.

7.8. Compactification of the Euclidean space. Prove that the Alexandroff one-point compactification of $\mathbb{R}^n$ is homeomorphic to S^n for all $n \geq 1$.

7.9. Hausdorff quotient. Let X be a topological space and let $\Delta := \{(x,y) \in X \times X \,:\, x = y\}$ be the diagonal of $X \times X$.

(i) Show that X is Hausdorff if and only in Δ is closed in $X \times X$.

(ii) Let $\sim$ be any equivalence relation on X and define $R := \{(x, y) \in X \times X \,:\, x \sim y\}$. Suppose that $q\colon X \to X/{\sim}$ is open. Show that $X/{\sim}$ is Hausdorff if and only if R is closed in $X \times X$.

7.10. Quotient with respect to a compact set ⚙. Let X be a Hausdorff space and let K be a compact subset of X. Show that:

(i) The quotient X/K is Hausdorff.

Note: The notation X/K is standard to indicate $X/{\sim}$, where the equivalence relation $\sim$ is given by $x \sim y$ if and only if $x = y$ or $x, y \in K$.

(ii) Let A be an open subset of X strictly contained in K (i.e. $A \subsetneq K$). Then the map $f\colon (X \setminus A)/(K \setminus A) \to X/K$ defined as $f([x]_A) := [x]$ is well-defined and a homeomorphism. Here we denote with $[x]_A$ and $[x]$ the equivalence classes in $(X \setminus A)/(K \setminus A)$ and X/K, respectively.

(iii) Item (ii) is not true if $A = K$.

(iv) Assume that X is compact. Then X/K is the Alexandroff one-point compactification of $X \setminus K$.