8. Homotopy and contractible spaces

Chef's table

In this problem set, we start working with the most basic notion in Algebraic Topology, that of *homotopy* (of maps, and topological spaces). The first seven exercises deal, one way or the other, with contractible spaces (those spaces that are equivalent to a point, so the *simplest ones* we can think of). They are all rather simple, except for 8.5 which is a bit more tedious, and also partly unrelated to the other ones (so maybe skip it at a first reading, and get back to it later). Through Problems 8.6 and 8.7 we introduce a basic operation (called *cone*), which produces a topological space $C(X)$ given a topological space X. As it often happens in Mathematics, the name is not random: if you take $X = S^1$ then $C(X)$ is an honest ice-cream cone.

Problem 8.8 is a first (and most important!) example of 'non-trivial' homotopy between maps on spheres, and it introduces some ideas that will come back later in my lectures, when I will prove tha so-called *hairy ball theorem*.

Lastly, Problem 8.9 and Problem 8.10 are meant to clearly suggest the difference between homotopies (of loops) fixing the basepoint and homotopies which are allowed to move basepoints around. The two notions are *heavily inequivalent* and these two exercises should guide you to build a space which exhibits this inequivalence. These outcome is conceptually very important, and (even if you can't solve 8.10) the result should be known and understood by all of you.

8.1. Contractible spaces are simply connected ⚙. Let X be a contractible space.

(i) Show that X is path-connected.

(ii) Prove that $\pi_1(X) \cong \{1\}$.

Note: Point (ii) *of this exercise follows directly from the homotopy invariance of the fundamental group. However, in this special case, the proof is much simpler.*

8.2. Functions on contractible spaces ✍. Prove the following statements.

(i) Let X be a path-connected topological space. Show X is contractible if and only if for any path-connected topological space Y and any pair of functions $f, g\colon X \to Y$, we have that f and g are homotopic.

(ii) Show that a topological space X is contractible if and only if for any topological space Y and any pair of continuous function $f, g\colon Y \to X$, we have that f and g are homotopic.

8.3. Homotopic paths ✍. Let X be a topological space, and let γ_1, γ_2 be paths in X with the same endpoints (i.e. $\gamma_1(0) = \gamma_2(0)$ and $\gamma_1(1) = \gamma_2(1)$). Show that γ_1 and γ_2 are homotopic if and only if there is a continuous map $F\colon D^2 \to X$ such that $F|_{\partial D^2}\colon S^1 \to X$ is a reparametrization of $\gamma_1 * \gamma_2^{-1}$.

8.4. Trivial fundamental group ✍. Let X be a path-connected topological space. Show that $\pi_1(X) = \{1\}$ if and only if for every pair of points x, y of X, there exists only one homotopy class of paths joining them.

8.5. Path of continuous functions ⚙. Let $I = [0,1]$, and let (X, d) be a metric space. Let $\gamma_0, \gamma_1\colon I \to X$ be paths in X with the same endpoints. Let $\mathcal{S}$ be the subset of $C(I, X)$ defined as $\mathcal{S} := \{f \in C(I, X) \,:\, f(0) = \gamma_0(0) = \gamma_1(0), f(1) = \gamma_0(1) = \gamma_1(1)\}$. Show that there is a continuous path $\Gamma\colon [0,1] \to \mathcal{S}$ between γ_0 and γ_1 (i.e. $\Gamma(0) = \gamma_0$, $\Gamma(1) = \gamma_1$) if and only if γ_0 and γ_1 are homotopic.

Note: Recall that $C(I, X) = \{f\colon I \to X \,:\, f$ is continuous$\}$ is a metric space if equipped with the distance $d(f, g) = \sup_{t\in[0,1]}\{d_X(f(t), g(t))\}$ (see Problem 6.1).

8.6. Cone that is not a topological manifold ✍. Find an example of a topological manifold X such that the cone $C(X) := (X \times [0,1])/(X \times \{0\})$ over X is not a topological manifold around $\bar{x} = [X \times \{0\}] \in C(X)$, i.e. the point $\bar{x}$ does not admit any neighborhood $U \subseteq C(X)$ of $\bar{x}$ homeomorphic to $\mathbb{R}^n$ for some $n \in \mathbb{N}$.

8.7. Every cone is contractible ⚙. Given a topological space X, let $C(X)$ denote the cone over X, i.e. $C(X) := (X \times [0,1])/(X \times \{0\})$. Show that

(i) $C(X)$ is path-connected;

(ii) $C(X)$ is contractible.

8.8. Homotopic maps on the sphere ⚙. Show that, if $n \in \mathbb{N}$ is odd, the antipodal map $-\mathrm{Id}_{S^n}\colon S^n \to S^n$ on the sphere is homotopic to the identity map Id_{S^n}.

8.9. Preparation to Problem 8.10 ⚙. Recall that a space X is contractible if there exists a continuous map $H\colon X \times [0,1] \to X$ and a point $x_0 \in X$ such that $H(x, 0) = x$ and $H(x, 1) = x_0$ for every $x \in X$. Note that we do not require that $H(x_0, t) = x_0$ for every possible $t \in [0,1]$. If we add this hypothesis, i.e. we ask $H(x_0, t) = x_0$ for all $t \in [0,1]$, then we say that X *deformation retracts* to a point (more precisely to the point x_0). The goal of this and the following exercises is to show that deformation retracting to a point is a stronger property than being contractible.

(i) Show that, if a space X deformation retracts to a point $x_0 \in X$, then for each neighborhood $U \subseteq X$ of x_0 there exists a neighborhood $V \subseteq U$ of x_0 such that the inclusion map $i\colon V \hookrightarrow U$ is homotopic in U to the constant map $c_{x_0}\colon V \to \{x_0\}$.

(ii) Let X be the subspace of $\mathbb{R}^2$ consisting of the horizontal segment $[0,1] \times \{0\}$ together with all the vertical segments $\{r\} \times [0, 1-r]$, for r a rational number in $[0,1]$. Show that the space X deformation retracts to any $x_0 \in [0,1] \times \{0\}$, but not to any other point.

8.10. Contractible spaces that do not deformation retract. Construct an example of a topological space X that is contractible but does not deformation retract to any point $x_0 \in X$.