9. Coverings

Chef's table

This problem set is totally about *playing with covering maps*. There are some examples to work out (specifically problems 9.1-9.2-9.3) and a few basic facts to prove (problems 9.4-9.5-9.6-9.7-9.8). Among these exercises (which are meant to be quite fast to solve, and basically only build on the very definition of covering map) I would like to stress the importance of Problem 9.7, which gives us an excuse to introduce the definition of *discrete subset* in a topological space (have a look at the note below and make sure you can follow the remarks in there). The fact that any fiber of a covering map is indeed a discrete subset is crucially employed in the proof of the monodromy theorem (Lecture 19) and, in a special case, already in computing the fundamental group of S^1 (Lecture 18). Problem 9.9 is a little lemma which extends the analogous statement for paths (i.e. that 'paths are evenly covered'), and is used in the proof of existence of lifts for homotopies. Last but not least, the challenge problem of this week is more accessible than other ones and is in fact an *excellent* way of testing your understanding of the ideas we have introduced in the last two weeks. This time I suggest that you all give it a look and try to think about it, as it may really clarify what Algebraic Topology is about.

9.1. Cover the circle ✍. Let $p\colon \mathbb{R} \to S^1$ be the map defined as

$$p(t) := (\cos(2\pi t), \sin(2\pi t)).$$

Show that p is a covering map (of infinite degree).

9.2. Cover the circle, reloaded ✍. Prove that $p\colon S^1 \to S^1$ given by $p(z) := z^n$ is a degree n covering map for any $n \geq 1$.

Note: Here we regard $S^1 \subseteq \mathbb{C}$ as unit circle.

9.3. Cover the punctured plane ✍. Prove that $p\colon \mathbb{C}^* \to \mathbb{C}^*$ given by $p(z) := z^n$ is a degree n covering map for any $n \geq 1$.

Note: Here we have denoted $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$.

9.4. Two criteria for covering maps ✍. Let $\tilde{X}, X$ be topological spaces and let $p\colon \tilde{X} \to X$ be a continuous function. Then:

(i) p is a covering map if and only if there is a cover of X consisting of evenly covered open sets;

(ii) p is a covering map if and only if there is a basis of X consisting of evenly covered open sets.

Deduce that a covering map is open.

9.5. Restrictions of covering maps ✍. Consider a covering map $p\colon \tilde{X} \to X$ and a subspace $A \subseteq X$. Defining $\tilde{A} := p^{-1}(A)$, show that the restriction $p|_{\tilde{A}}\colon \tilde{A} \to A$ is a covering map.

9.6. Composition of covering maps ✍. Let X, Y, Z be topological spaces, and let $p\colon X \to Y$ and $q\colon Y \to Z$ be covering maps. Assume that $q^{-1}(z)$ is a finite set for all $z \in Z$. Show that $q \circ p\colon X \to Z$ is a covering map.

9.7. Fibers are discrete ⚙. Let $p\colon \tilde{X} \to X$ be a covering map, with X a Hausdorff space. Show that for any $x \in X$ the fiber $p^{-1}(x)$ is a discrete subset of $\tilde{X}$. Deduce that, if $\tilde{X}$ is compact and X is connected, then the covering in question has finite degree.

Note: We say that a subset A of a topological space X is discrete if for all $x \in X$ there exists a neighborhood $U = U(x)$ such that $(U \setminus \{x\}) \cap A = \emptyset$. Observe that:

- *If $A \subseteq X$ is a discrete subset, then the induced topology on A as a subspace of X is discrete.*
- *The converse of the previous assertion is false, for instance $A := \{1/n : n \in \mathbb{N}_*\}$ inherits the discrete topology as a subspace of $\mathbb{R}$ (the standard real line) but it is not a discrete subset.*

Note that, as a result of the first item above, a continuous map $f\colon Y \to X$ with Y connected, such that $f(Y) \subseteq A$ with A discrete is actually constant.

9.8. When the fibers are finite ⚙. Let $p\colon \tilde{X} \to X$ be a covering map with $p^{-1}(x)$ finite and non-empty for all $x \in X$. Show that $\tilde{X}$ is compact Hausdorff if and only if X is compact Hausdorff.

9.9. Homotopies are evenly covered ⚙. Let $p\colon X \to Y$ be a covering map, and let $F\colon [0,1] \times [0,1] \to Y$ be a homotopy between two paths. For each $y \in Y$, let U_y be an evenly covered neighborhood of y. Show that there is $n > 0$ such that, subdividing $[0,1] \times [0,1]$ is squares of side length $1/n$, we obtained that the image under F of every such sub-square is contained in U_y for some $y \in Y$.

9.10. Complements of circles and lines ⊛. Let $\gamma := \{(x,y,z) \in \mathbb{R}^3 : x^2 + y^2 = 1, z = 0\}$ and let $r := \{(x,y,z) \in \mathbb{R}^3 : x = y = 0\}$. Prove that the spaces $\mathbb{R}^3 \setminus \gamma$ and $\mathbb{R}^3 \setminus (\gamma \cup r)$ are not homeomorphic.