10. Computing the fundamental group - Part I

Chef's table

This week we start computing fundamental group, mainly using Van Kampen's Theorem (but possibly in combination with other tools). The first two problems are sort of basic warm-up exercises to get a feeling for the subject. Problems 10.3 - 10.4 - 10.5 are almost identical, and they all build on the same trick (think in terms of the planar models!); you can write down the solution to just one of them, but make sure you give some thought about all. From there (so building on these results), using Van Kampen you can compute the fundamental group of the torus (which we knew anyway, but through a different method), of the Klein bottle and of higher-genus surfaces. Problem 10.8 is the most important in this series, and learning this trick will trivialise half of the problems on this subject (see Problem 10.9 for a first, striking, application); writing down all homotopies of 10.8 explicitly might be a bit tedious, so just make sure to have a clear picture (and keep in mind this result for the future).

10.1. Topological manifold minus a point ✍. Let X be a connected topological manifold, of dimension $n \geq 3$. Prove that for every $x \in X$ one has $\pi_1(X) \cong \pi_1(X \setminus \{x\})$.

10.2. Plane without the circle ✍. Show that there is no homeomorphism $f\colon \mathbb{R}^2 \setminus S^1 \to \mathbb{R}^2 \setminus S^1$ such that $f(0,0) = (2,0)$.

10.3. Torus minus a point ✍. Compute the fundamental group of $T^2 \setminus \{p\}$, where T^2 stands for the standard torus and p is any point in T^2.

10.4. Klein bottle minus a point ✍. Compute the fundamental group of $K^2 \setminus \{p\}$, where K^2 stands for the Klein bottle and p is any point in K^2.

10.5. Surface minus a point ⚙. For any $g \geq 2$, let Σ_g denote a genus g closed orientable surface. Compute the fundamental group of $\Sigma_g \setminus \{p\}$.

Note: It may be useful to use the following definition of the genus g surface Σ_g. Let P_{4g} be a (say, regular) $4g$-sided polygon, whose sides are enumerated from 0 to $4g-1$ in clockwise order. Then Σ_g is the surface obtain from P_{4g} identifying the side S parametrized in clockwise direction with the side $S+2$ parametrized in counterclockwise direction, for all $S = 0, 4, 8, \ldots, 4(g-1)$ and for all $S = 1, 5, 9, \ldots, 4(g-1)+1$.

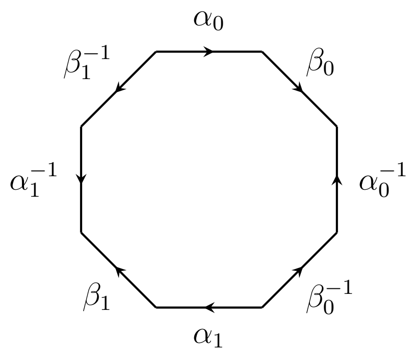

Figure 1: Example of the construction of Σ_g by identification of the sides of a $4g$-sided polygon in the case $g = 2$.

10.6. Application of Van Kampen's Theorem. Relying on the result of Problems 10.3 and 10.4, employ Van Kampen's Theorem to compute $\pi_1(T^2)$ and $\pi_1(K^2)$.

10.7. Fundamental group of a surface. Relying on the result of Problem 10.5, employ Van Kampen's Theorem to compute $\pi_1(\Sigma_g)$.

10.8. Bouquet of circles. Prove that the following three topological spaces are homotopy equivalent:

- $X_1 :=$ the wedge sum of k circles;
- $X_2 := \mathbb{R}^2$ minus k points;
- $X_3 :=$ the circle union with k of its radii.

Hence compute their fundamental group.

10.9. Euclidean space minus a finite number of points. Compute the fundamental group of $\mathbb{R}^n$ minus k points, as one varies $n \geq 2$ and $k \geq 0$. In particular, determine all pairs (n, k) so that such space is simply connected.

10.10. Edges of the cube. Compute the fundamental group of the space X that is the union of the edges of the cube $C = [0, 1]^3$. Moreover compute the fundamental group of $X \setminus \{x\}$ as one varies $x \in X$.