11. Computing fundamental group - Part II

Chef's table

In this (second to last) problem set, we keep training on themes related to the fundamental group of a topological space. The first two problems are two (very instructive!) exercises on the correspondence between Algebra (structure of free groups) on one side and Topology (structure of wedges) on the other side. It is always helpful, both here and at a more advanced level, to reinterpret facts on either side of the mirror. Problems from 11.3 to 11.8 are six exercises where you ultimately need to combine suitable retractions (to reduce those spaces to simpler ones) and, possibly, Van Kampen's Theorem applied to correctly chosen sets (the art is always to pick the correct sets, which makes the whole difference between a straightforward conclusion and an insane mess). Problem 11.9 provides a topological proof of the Fundamental Theorem of Algebra (can you list 5 different proofs of this result?). Lastly, Problem 11.10 is... quite a challenge! Try it at your own risk.

11.1. Finite order words ✍. Determine all and only elements of finite order in the free group F_n with $n \geq 1$ generators.

11.2. Non-isomorphic free groups ✍.

(i) Prove, through a topological argument, that the wedge of m circles is not homeomorphic to the wedge of n circles if $m \neq n$.

(ii) Prove that the group F_m is not isomorphic to F_n if $m \neq n$, hence deduce that the wedge of m circles is not homotopic to the wedge of n circles if $m \neq n$.

11.3. Plane and circle ✍. Compute the fundamental group of the following subspaces of $\mathbb{R}^3$:

(i) $X_1 := \{(0, y, z) \in \mathbb{R}^3\} \cup \{(x, y, 0) \in \mathbb{R}^3 : x^2 + y^2 = 1,\ x \geq 0\}$, i.e. the union of a plane with a half-circle with end points on the plane;

(ii) $X_2 := \{(0, y, z) \in \mathbb{R}^3\} \cup \{(x, y, 0) \in \mathbb{R}^3 : x^2 + y^2 = 1\}$, i.e. the union of a plane with a circle that intersects the plane transversely in two points.

11.4. Difference of linear spaces ✍. Compute the fundamental group of $V \setminus W$, where V is a linear space over $\mathbb{R}$ of dimension $n \geq 2$ and $W \subset V$ is a linear subspace of dimension k with $0 \leq k \leq n - 2$.

11.5. Intersecting planes ⚙. Let X be the topological subspace of $\mathbb{R}^3$ given by the union of three distinct planes (not necessarily passing through the origin). Compute the fundamental group in any configuration such that X is connected.

11.6. Union of sphere and planes ⚙. Compute $\pi_1(X)$, where X is the subspace of $\mathbb{R}^3$ obtained as the union of the unit sphere and the three coordinate planes, namely

$$X := S^2 \cup \{x = 0\} \cup \{y = 0\} \cup \{z = 0\}.$$

11.7. Complement of two linked circles. Compute the fundamental group of the complement of two linked circles in the unit sphere $S^3 \subseteq \mathbb{R}^4$. Same question for $\mathbb{R}^3$ in lieu of S^3.

11.8. Subspace of the two-dimensional complex space. Compute the fundamental group of the subspace X of $\mathbb{C}^2$ defined as

$$X := \{(z, w) \in \mathbb{C}^2 \,:\, |z| = |w| = 1\} \cup \{(1, w) \in \mathbb{C}^2\}.$$

11.9. Fundamental Theorem of Algebra. Let $p(z) = z^n + a_{n-1}z^{n-1} + \ldots + a_0$ with $n > 0$ be a complex polynomial with positive degree. We want to show that p admits at least one root.

(i) Let us first assume that $\|a_0\| + \ldots + \|a_{n-1}\| < 1$.

 (a) Let $f\colon S^1 \to \mathbb{C}^*$ be the map defined as $f(z) := z^n$. Show that f is not null-homotopic.

 (b) Show that there is a homotopy H between f and the restriction of p on S^1, such that H has values in $\mathbb{C}^*$.

 (c) Assume, by contradiction, that p does not admit any root. Show that this implies that p is null-homotopic in $\mathbb{C}^*$ and obtain a contradiction.

(ii) Reduce the general case to the case when $\|a_0\| + \ldots + \|a_{n-1}\| < 1$ and prove the Fundamental Theorem of Algebra.

11.10. Complex lines through the origin. Compute the fundamental group of $\mathbb{C}^2 \setminus (\ell_1 \cup \ell_2)$ where ℓ_1, ℓ_2 are two distinct complex lines through the origin.