# 12. Coverings, reloaded

### Chef's table

This final problem set is centered around the interplay of coverings and fundamental group. It is a good review of many/most of the techniques we have seen along the semester, and you will have two weeks to solve it.

Some of the exercises are centered around the construction of simply-connected covering spaces (i.e., of the universal cover of a given topological space): Problem 12.5 revisits one key technical point of the general construction; Problem 12.3 (but also, from different perspectives, Problem 12.4, and the challenge Problem 12.10) shows that in concrete cases one can *forget about the abstract proof and perform these constructions with bare hands*; while Problem 12.6 shows that if the base space $X$ is locally too wild then the universal cover just does not exist (hence one cannot really weaken the assumptions in our theorem).

(The second part of) Problem 12.7 is a famous (and somewhat surprising!) result in Topology: *on the Earth there are always two antipodal locations having the same temperature and pressure.* This theorem is proven by lifting the right curves. Problem 12.8 is also a famous result, which follows straight from an application of 12.7 by simply choosing the right function $f$. Lastly, Problem 12.9 is what I would call an 'exercise of style', which concerns calculating, with two different techniques (either using Van Kampen or using pure covering arguments), the fundamental group of a certain space: I find it very instructive in that it sheds some light on the strengths and weaknesses of these two methods.

**12.1. Compact universal cover and functions to the circle ✍.** Let $X$ be a (locally path-connected) topological space having compact universal cover. Prove that any continuous function $f\colon X \to S^1$ is homotopic to a constant.

**12.2. Compact universal cover and other coverings ✍.** Prove that a topological space having compact universal cover admits finitely many covering spaces up to isomorphism.

**12.3. Coverings of the torus ⚙.** Let $T^2$ be the standard torus. For every subgroup $H$ of $\pi_1(T^2)$, find a covering map $q\colon X \to T^2$ such that $\mathrm{Im}(q_*) = H$.

*Hint: Note that, since $\pi_1(T^2) = \mathbb{Z} \times \mathbb{Z}$ is an abelian group, every covering space of $T^2$ is normal.*

**12.4. Different coverings of the torus ✍.** Determine two covering maps $p\colon X \to T^2$ and $p'\colon X' \to T^2$ of the standard torus $T^2$ such that:

- $p\colon X \to T^2$ and $p'\colon X' \to T^2$ have the same (finite) number of sheets;

- there do not exist homeomorphisms $\phi\colon X \to X'$ and $\psi\colon T^2 \to T^2$ with $p' \circ \phi = \psi \circ p$.

**12.5. Construction of the universal cover - technical details.** In this problem we wish to revisit one technical point in the construction of the universal cover of a topological space. Let then $X$ be a topological space (assumed to be path-connected, locally path-connected and semi-locally simply-connected), and fix $x_0 \in X$. In class (Lecture 24) we have defined a simply-connected topological space $\tilde{X}$ by declaring

$$\tilde{X} := \{[\gamma] \,:\, \gamma\colon I \to X \text{ satisfies } \gamma(0) = x_0\},$$

where $[\gamma]$ denotes the equivalence class under homotopies preserving both endpoints; we also defined the surjective map $p\colon \tilde{X} \to X$ by $p([\gamma]) := \gamma(1)$. Check that $p$ is continuous, and then that $p$ is in fact a covering map.

**12.6. When the universal cover does not exist.** Let $X$ be the subspace of $\mathbb{R}^2$ consisting of the four sides of the square $[0,1] \times [0,1]$ together with the segments of the vertical lines $x = 1/2, 1/3, 1/4, \ldots$ inside the square. Show that for every covering space $p\colon \tilde{X} \to X$ there is some neighborhood of the left edge $\{0\} \times [0,1]$ of $X$ that lifts homeomorphically to $\tilde{X}$. Deduce that $X$ has no simply-connected covering space.

**12.7. Functions on antipodal points.** Prove the following versions of the Borsuk-Ulam theorem.

(i) Prove that any continuous map $f\colon S^2 \to \mathbb{R}$ must attain the same value at a pair of antipodal points.

(ii) Prove that any continuous map $f\colon S^2 \to \mathbb{R}^2$ must attain the same value at a pair of antipodal points.

**12.8. Covering of the sphere with closed sets.** Let $A_1, \ldots, A_k$ be closed subsets of $S^2$ whose union is $S^2$ itself. Prove that if $k \leq 3$ then there exists $i \in \{1, \ldots, k\}$ such that $A_i$ contains a pair of antipodal points. How about $k \geq 4$?

**12.9. Projective space minus one point.** Compute, in two different ways (directly via Van Kampen's Theorem, and alternatively using covering arguments) the fundamental group of $\mathbb{P}^2(\mathbb{R})$ minus one point.

**12.10. Explicit construction of universal covers.** Construct a simply-connected covering space of the space $X \subseteq \mathbb{R}^3$ that is the union of a sphere and a diameter. Do the same when $X$ is the union of a sphere and a circle intersecting it in two points.