1. Topological spaces and continuous functions

1.1. Question with four points. Let $X = \{a, b, c, d\}$ a set with four points. Which of the following ones are topologies for X?

(i) $\{\emptyset, X, \{a\}, \{b\}, \{a, c\}, \{a, b, c\}, \{a, b\}\}$

(ii) $\{\emptyset, X, \{a\}, \{b\}, \{a, b\}, \{b, d\}\}$

(iii) $\{\emptyset, X, \{a, c, d\}, \{b, c, d\}\}$

1.2. The arrow line. For each $x \in \mathbb{R}$, let $I_x = (x, \infty)$, and let $I_\infty = \emptyset$ and $I_{-\infty} = \mathbb{R}$. Check that

$$\mathcal{T} = \{I_x \mid x \in \mathbb{R} \cup \{-\infty, \infty\}\}$$

defines a topology on $\mathbb{R}$.

1.3. Point and finite complements. Let X be a set and let p be an element of X. Check that

$$\mathcal{T} = \{A \subseteq X \mid p \notin A \text{ or } X \setminus A \text{ is finite}\}$$

defines a topology on X.

1.4. Continuous function. Let $\mathcal{T}$ be the topology for $\mathbb{R}$ described in Problem 1.2. Which of the following functions $f : \mathbb{R} \to \mathbb{R}$ are continuous *with respect to* $\mathcal{T}$?

(i) $f(x) = x^2$

(ii) $f(x) = x^3$

(iii) $f(x) = \begin{cases} 5 & \text{if } x > 5 \\ 0 & \text{otherwise} \end{cases}$

(iv) $f(x) = -x$

1.5. Elaborate on Problem 1.4. In this exercise, we want to understand a little bit better continuous maps in the topology of Problem 1.2. For the sake of this exercise, we say that a map $f : \mathbb{R} \to \mathbb{R}$ is *standard-continuous*, if it is continuous with respect to the usual topology on $\mathbb{R}$. We say that it is $\mathcal{T}$*-continuous* if it is continuous with respect to the topology described in Problem 1.2. Let f be a function that is standard-continuous. Prove that f is $\mathcal{T}$-continuous if and only if it is monotonically increasing.

1.6. Definitions of interior. The goal of this exercise is to give some equivalent characterizations for the *interior* of a set. Let X be a topological space and let Y be a subset of X. Moreover, let:

(i) $\operatorname{int}(Y) = \{x \in X \mid \text{ there exists } O \text{ open such that } x \in O \subseteq Y\}$;

(ii) Y_1 be the maximal open set that is contained in Y (if it exists);

(iii) Y_2 be the union of all the open sets that are contained in Y.

Show that Y_1 exists and $\operatorname{int}(Y) = Y_1 = Y_2$.

1.7. Definitions of closure ✍. The goal of this exercise is to give some equivalent characterizations for the *closure* of a set. Let X be a topological space and let Y be a subset of X. Let:

(i) $\overline{Y} = \operatorname{int}(Y) \cup \{x \in X \mid \text{ for each open } O \ni x,\, O \cap Y \neq \emptyset \neq O \cap (X \setminus Y)\}$;

(ii) Y_1 be the minimal closed set that contains Y (if it exists);

(iii) Y_2 be the intersection of all the closed sets that contain Y;

(iv) $Y_3 = X \setminus \operatorname{int}(X \setminus Y)$.

Show that Y_1 exists and $\overline{Y} = Y_1 = Y_2 = Y_3$.

1.8. Examples with closure ⚙. Give an example of two subsets A and B of $\mathbb{R}$ such that:

$$A \cap B = \emptyset, \qquad \overline{A} \cap B \neq \emptyset, \qquad A \cap \overline{B} \neq \emptyset.$$

Bonus: can you find two (essentially different) such examples?

1.9. Union and closure ⚙. Let A and B be subsets of a topological space X. Show that:

(i) $\operatorname{int}(A) \cap \operatorname{int}(B) = \operatorname{int}(A \cap B)$;

(ii) $\operatorname{int}(A) \cup \operatorname{int}(B) \subseteq \operatorname{int}(A \cup B)$;

(iii) $\overline{A} \cup \overline{B} = \overline{A \cup B}$;

(iv) $\overline{A \cap B} \subseteq \overline{A} \cap \overline{B}$;

(v) Give one example where the equality in part (ii) is satisfied, one where it fails, one where the equality in part (iv) is satisfied and one where it fails.

1.10. Topology with many open sets ⬡. Prove that a topology on $\{1, 2, \ldots, n\}$ with k open sets cannot exist if $3 \cdot 2^{n-2} < k < 2^n$.

1. Solutions

Solution of 1.1:

(i) Yes (it is straightforward to check).

(ii) No, since for example $\{a\} \cup \{b, d\} = \{a, b, d\}$ is not included in the set.

(iii) No, since for example $\{a, c, d\} \cap \{b, c, d\} = \{c, d\}$ is not included in the set.

Solution of 1.2: First observe that $\emptyset = I_\infty, \mathbb{R} = I_{-\infty} \in \mathcal{T}$. Hence it is sufficient to check that $\mathcal{T}$ is closed under finite intersection and countable union.

Given I_x and I_y such that (without loss of generality) $x \leq y$, then $I_x \cup I_y = I_x$ and $I_x \cap I_y = I_y$. Moreover, let $\{x_i\}_{i\in I}$ be a (possibly infinite) family of elements of $\mathbb{R}$, and let $\hat{x} = \inf_{i\in I} x_i$. Then we have

$$\bigcup_{i\in I} I_{x_i} = I_{\hat{x}}.$$

Indeed, for each $y \in I_{\hat{x}}$ (that is $y > \hat{x}$), by definition of infimum there exists $j \in I$ such that $x_j < y$, i.e. $y \in I_{x_j}$, and hence $I_{\hat{x}} \subseteq \bigcup_{i\in I} I_{x_i}$. The other inclusion is obvious since $I_{x_i} \subset I_{\hat{x}}$ for every $i \in I$.

Solution of 1.3: We start by noticing that $p \notin \emptyset$ and that $X \setminus X = \emptyset$ is finite, thus $\emptyset, X \in \mathcal{T}$. We need to check that $\mathcal{T}$ is closed under finite intersection and (possibly infinite) union. For the intersection, let A_1 and A_2 be elements of $\mathcal{T}$. If at least one of them does not contain p, then the intersection $A_1 \cap A_2$ also does not contain p and so it is an element of $\mathcal{T}$. Thus, let us assume that A_1 and A_2 both contain p, which means that the sets $X \setminus A_1$ and $X \setminus A_2$ are both finite. Note that

$$X \setminus (A_1 \cap A_2) = (X \setminus A_1) \cup (X \setminus A_2).$$

Since $X \setminus A_1$ and $X \setminus A_2$ are finite, so is their union $X \setminus (A_1 \cup A_2)$, which proves that $A_1 \cap A_2$ is contained in $\mathcal{T}$.

Now, let $\{A_i\}_{i\in I}$ be a (possibly infinite) family of elements of $\mathcal{T}$, and let $A = \bigcup_{i\in I} A_i$ be their union. If for all $i \in I$ the element p is not contained in A_i, then $p \notin A$, and thus $A \in \mathcal{T}$. Conversely, suppose that there is $j \in I$ such that $p \in A_j$ and therefore $X \setminus A_j$ is finite, because $A_j \in \mathcal{T}$. Since

$$X \setminus \left(\bigcup_{i\in I} A_i \right) = \bigcap_{i\in I} (X \setminus A_i),$$

and we know that at least one element on the right hand side is finite (the one corresponding to j), we get the claim.

Solution of 1.4:

(i) No, since for example $f^{-1}(I_1) = (-\infty, -1) \cup (1, \infty) \notin \mathcal{T}$.

(ii) Yes, since $f^{-1}(I_x) = I_{\sqrt[3]{x}}$ for all $x \in \mathbb{R}$ and $f^{-1}(I_{\pm\infty}) = I_{\pm\infty}$.

(iii) Yes, since

$$f^{-1}(I_x) = \begin{cases} \emptyset & \text{if } x \geq 5 \\ I_5 & \text{if } 0 \leq x < 5 \\ \mathbb{R} = I_{-\infty} & \text{if } x < 0. \end{cases}$$

(iv) No, since for example $f^{-1}(I_1) = (-\infty, -1) \notin \mathcal{T}$.

Solution of 1.5: First let us prove that if f is monotonically increasing then it is $\mathcal{T}$-continuous, i.e. $f^{-1}(I_a) \in \mathcal{T}$ for every $a \in \mathbb{R} \cup \{-\infty, \infty\}$. Since I_a is standard-open, then $f^{-1}(I_a)$ is standard-open. Moreover observe that, if $x \in f^{-1}(I_a)$ and $y \geq x$, then $y \in f^{-1}(I_a)$. Indeed $f(y) \geq f(x) > a$ by monotonicity of f. As a result, $f^{-1}(I_a)$ is of the form $(b, \infty) = I_b$ for some $b \in \mathbb{R} \cup \{-\infty, \infty\}$, as we wanted.

Let us now prove the other implication, namely that if f is $\mathcal{T}$-continuous then it is monotonically increasing. Assume by contradiction that f is not monotonically increasing, then there exist $x \leq y$ such that $f(x) > f(y)$, i.e. $x \in f^{-1}(I_{f(y)})$. However, since f is $\mathcal{T}$-continuous, $f^{-1}(I_{f(y)})$ is of the form $I_b = (b, \infty)$ for some $b \in \mathbb{R} \cup \{-\infty, \infty\}$. Note that $b < x \leq y$, hence $y \in I_b = f^{-1}(I_{f(y)})$, i.e. $f(y) > f(y)$, which is patently a contradiction.

Solution of 1.6: First observe that Y_2 is an open set contained in Y and it contains any other open set contained in Y. Hence a maximal open set Y_1 in Y exists and coincide with Y_2. Hence, we just need to prove that $\text{int}(Y)$ coincides as well with $Y_1 = Y_2$.

Take $x \in Y_1 = Y_2$, then x is contained in an open set contained in Y, since $Y_1 = Y_2$ is open. Hence $x \in \text{int}(Y)$. Viceverse, let $x \in \text{int}(Y)$. Then there exists O such that $x \in O \subseteq Y$. By definition, $O \subseteq Y_1 = Y_2$, and thus $x \in Y_1 = Y_2$.

Solution of 1.7: First observe that Y_2 is a closed set containing Y and it is contained in any other closed set containing Y. Hence a minimal closed set containing Y exists and coincide with Y_2. We will know show that both $\overline{Y}$ and $Y_1 = Y_2$ coincide with Y_3.

Proof of $\overline{Y} = Y_3$. Let x be a point of X, then exactly one of the following three possibilities holds:

(1) There exists an open set O such that $x \in O \subseteq Y$;

(2) There exists an open set O such that $x \in O \subseteq X \setminus Y$;

(3) For every open set O that contains x, O intersects both Y and $X \setminus Y$.

We defined $\overline{Y}$ to be the set of points that satisfy either (1) or (3). Moreover, the set of points that satisfy (2) is defined to be $\text{int}(X \setminus Y)$. Thus $\overline{Y} = X \setminus \text{int}(X \setminus Y) = Y_3$.

Proof of $Y_1 = Y_2 = Y_3$. If Y_1 is the minimal closed set that contains Y, then $X \setminus Y_1$ is the maximal open set that does not contain Y. In particular, $X \setminus Y_1$ is the maximal open set that is contained in $X \setminus Y$. By the previous exercise $X \setminus Y_1 = \operatorname{int}(X \setminus Y)$, which is equivalent to the desired equality.

Solution of 1.8: It holds that $\overline{\mathbb{Q}} = \overline{\mathbb{R} \setminus \mathbb{Q}} = \mathbb{R}$. Thus $A = \mathbb{Q}$ and $B = \mathbb{R} \setminus \mathbb{Q}$ works. Another example is $A = [0,1) \cup [2,3)$ and $B = [1,2)$.

Solution of 1.9: Let us prove the five statements separately.

(i) The sets $\operatorname{int}(A)$ and $\operatorname{int}(B)$ are the maximal open sets contained in A and B respectively. Hence $\operatorname{int}(A) \cap \operatorname{int}(B)$ is an open set contained in $A \cap B$, which implies that $\operatorname{int}(A) \cap \operatorname{int}(B) \subset \operatorname{int}(A \cap B)$. On the other hand, $\operatorname{int}(A \cap B)$ is the maximal open set contained in $A \cap B$. Therefore $\operatorname{int}(A \cap B)$ is an open set contained in both A and B, which proves that $\operatorname{int}(A \cap B)$ is contained also in $\operatorname{int}(A)$ and $\operatorname{int}(B)$ and thus in their intersection.

(ii) The union $\operatorname{int}(A) \cup \operatorname{int}(B)$ is an open set contained in $A \cup B$, therefore it is contained also in $\operatorname{int}(A \cup B)$.

(iii) Given a subset $C \subset X$, let us denote by $C^{\mathsf{c}} := X \setminus C$ its complement in X. Note that $\overline{C} = X \setminus \operatorname{int}(C^{\mathsf{c}})$ (which we used also in the solution of Problem 1.8). Then the result follows from point (i) since

$$\begin{aligned}\overline{A \cup B} &= X \setminus \operatorname{int}((A \cup B)^{\mathsf{c}}) = X \setminus \operatorname{int}(A^{\mathsf{c}} \cap B^{\mathsf{c}}) = X \setminus (\operatorname{int}(A^{\mathsf{c}}) \cap \operatorname{int}(B^{\mathsf{c}})) \\ &= (X \setminus \operatorname{int}(A^{\mathsf{c}})) \cup (X \setminus \operatorname{int}(B^{\mathsf{c}})) = \overline{A} \cup \overline{B}.\end{aligned}$$

(iv) Since $A \cap B \subseteq A$ and $A \cap B \subseteq B$, we have that $\overline{A \cap B} \subseteq \overline{A}$ and $\overline{A \cap B} \subseteq \overline{B}$. Thus the conclusion follows.

(v) The equalities in (ii) and (iv) are trivially satisfied if $A = B$.

An example where equality (ii) fails is when $A = [0,1]$ and $B = [1,2]$. Indeed $\operatorname{int}(A \cup B) = (0,2)$ but $\operatorname{int}(A) \cup \operatorname{int}(B) = (0,1) \cup (1,2)$.

An example where equality (iv) fails is when $A = (0,1)$ and $B = (1,2)$. Indeed $A \cap B = \emptyset$, but $\overline{A} \cap \overline{B} = \{1\}$.

Solution of 1.10: