2. Bases, subspaces and products

2.1. Homeomorphisms ✍. Show that the following topological spaces are homeomorphic.

(i) The interval $[0,1]$ and the interval $[2,5]$.

(ii) The interval $(-1,1)$ and the real line $\mathbb{R}$.

(iii) The closed disk of radius one in $\mathbb{R}^2$ and the closed square $[-1,1]\times[-1,1]$ in $\mathbb{R}^2$.

2.2. Existence of infinitely many primes ⚙. Let $\mathbb{Z}$ be the set of integer numbers. For every pair of integers $a,b\in\mathbb{Z}$ with $b>0$, let $B_{a,b}$ be the set

$$B_{a,b} \coloneqq \{a+kb \,:\, k\in\mathbb{Z}\}.$$

Prove the following facts:

(i) The set $\mathcal{B}=\{B_{a,b} \,:\, a,b\in\mathbb{Z}, b>0\}$ forms a basis for a topology $\mathcal{T}$ on $\mathbb{Z}$.

(ii) For every $a,b\in\mathbb{Z}$ with $b>0$, the set $B_{a,b}$ is both open and closed in $\mathbb{Z}$ with respect to $\mathcal{B}$.

(iii) Let $P=\{2,3,5,\dots\}$ be the set of prime numbers. Use the above facts to show that P needs to be infinite. *Hint: Consider the set* $\mathbb{Z}\setminus\bigcup_{p\in P}B_{0,p}$.

2.3. Non-standard topology ⚙. Let $\mathcal{B}$ be the following family of subsets of $\mathbb{R}$:

$$\mathcal{B}\coloneqq\{[a,b) \,:\, a\in\mathbb{Z}, b\in\mathbb{R}, a<b\}.$$

(i) Prove that there exists a topology $\mathcal{T}$ for which $\mathcal{B}$ is a basis.

(ii) Determine the interior and the closure of $A=(1/2,2)$ in the topology $\mathcal{T}$.

2.4. Product of maps ✍. Let X,Y,Z,W be topological spaces. Given two functions $f\colon X\to Z$ and $g\colon Y\to W$, we can define their product map $(f\times g)\colon X\times Y\to Z\times W$ as $(f\times g)(x,y)\coloneqq(f(x),g(y))$, which is continuous if and only if f and g are continuous.

(i) Show that if f and g are open then so is $f\times g$.

(ii) Show with a counterexample that the product of closed functions is not necessarily closed.

Note: We say that a function $f\colon X\to Z$ is open *if for every open set $O\subseteq X$ we have that $f(O)$ is open. Similarly f is* closed *if the image of each closed set is closed.*

2.5. Interior and closure of a product ⚙. Let X and Y be topological spaces, and let A,B be subsets of X,Y respectively. Show that

(i) $\operatorname{int}(A)\times\operatorname{int}(B)=\operatorname{int}(A\times B)$.

(ii) $\overline{A} \times \overline{B} = \overline{A \times B}$.

2.6. Sub-subspaces ✍. Let X be a topological space equipped with a topology $\mathcal{T}_X$, and let Y be a subset of X with the subset topology $\mathcal{T}_Y$ induced by $\mathcal{T}_X$. Given a subset Z of Y, we can consider on it the subset topologies $\mathcal{T}_{Z,Y}$ and $\mathcal{T}_{Z,X}$ induced by $\mathcal{T}_Y$ and $\mathcal{T}_X$, respectively. Show that $\mathcal{T}_{Z,Y} = \mathcal{T}_{Z,X}$.

2.7. Interior of subspaces ✍. Let Y be a subspace of a topological space X (i.e. Y is a topological space equipped with the subspace topology) and let A be a subset of Y. Defining $\operatorname{int}_X(A)$, $\operatorname{int}_Y(A)$ as the interiors of A with respect to X and Y respectively, show that $\operatorname{int}_X(A) \subseteq \operatorname{int}_Y(A)$ and give an example where the equality does not hold.

2.8. Finite metric space ✍. Let (X, d) be a metric space consisting of a finite number of points. Show that in X the distance topology coincides with the discrete topology.

2.9. p-adic numbers ⚙. Let p be a prime number and $d_p\colon \mathbb{Z} \times \mathbb{Z} \to [0, \infty)$ be the function defined by

$$d_p(x, y) := p^{-\max\{m \in \mathbb{N} \,:\, p^m | x-y\}},$$

where we understand $p^{-\max\{m \in \mathbb{N}\}}$ to be 0. Prove that d_p is a metric on $\mathbb{Z}$ and that $d_p(x, y) \leq \max\{d_p(x, z), d_p(z, y)\}$ for every $x, y, z \in \mathbb{Z}$.

Note: Here $p^m \mid x - y$ means that p^m divides $x - y$.

2.10. Open sets with the same boundary ⊛. For which $n, k \in \mathbb{N}_* = \{1, 2, 3, \ldots\}$ do there exist k distinct open subsets of $\mathbb{R}^n$ with the same boundary?

2. Solutions

Solution of 2.1:

(i) Consider the map $f\colon [0,1] \to [2,5]$ defined as $f(t) \coloneqq 2 + 3t$. It is clear that f is continuous and bijective with continuous inverse. Indeed, we have $f^{-1}(t) = \frac{t-2}{3}$. Therefore f is a homeomorphism.

(ii) Consider the map $f\colon (-1,1) \to \mathbb{R}$ defined as

$$f(t) = \begin{cases} \frac{t}{1-t} & \text{if } t \geq 0 \\ \frac{t}{1+t} & \text{if } t < 0. \end{cases}$$

Since $t/(1-t) = t/(1+t)$ if $t = 0$, the map is continuous. Moreover $\lim_{t\to\pm 1} f(t) = \pm\infty$ and f is strictly increasing, since $f'(t)$ is well-defined and greater than 0 for $t \in (-1,0) \cup (0,1)$. Therefore f is bijective with inverse

$$f^{-1}(t) = \begin{cases} \frac{t}{1+t} & \text{if } t \geq 0 \\ \frac{t}{1-t} & \text{if } t < 0, \end{cases}$$

which is a continuous function. Thus f is a homeomorphism.

Alternative solution (sketch): First note that $(-1,1)$ is homeomorphic to $(-\pi/2, \pi/2)$ via the map $f(t) \coloneqq \pi t/2$. Then the interval $(-\pi/2, \pi/2)$ is homeomorphic to $\mathbb{R}$ via the tangent function $\tan : (-\pi/2, \pi/2) \to \mathbb{R}$, which you should know (or check) that is a homeomorphism.

(iii) We want to produce a homeomorphism from the closed unit disk $B = \{x \in \mathbb{R}^2 : |x - (0,0)| \leq 1\}$ to the square $Q = [-1,1] \times [-1,1]$. Let us consider polar coordinates $(\rho, \theta) \in [0,\infty) \times [0, 2\pi)$ on $\mathbb{R}^2$. Note that, for each value of $\theta \in [0, 2\pi)$, there is exactly one value $\rho_B(\theta) \in (0,\infty)$ such that $(\rho_B(\theta), \theta) \in \partial B$ (i.e. $\rho_B(\theta) = 1$), and exactly one value $\rho_Q(\theta) \in (0,\infty)$ such that $(\rho_Q(\theta), \theta) \in \partial Q$. Let $g : [0, 2\pi) \to (0,\infty)$ be the function defined as $g(\theta) \coloneqq \rho_Q(\theta)/\rho_B(\theta) = \rho_Q(\theta)$, which is easily continuous (it can be written explicitly, checking the continuity). Then, let $f\colon B \to Q$ be the function defined as $f(\rho, \theta) \coloneqq (g(\theta)\rho, \theta)$. The function f is a bijection from B to Q with inverse $f^{-1}(\rho, \theta) = (\rho/g(\theta), \theta)$ (note that g is always positive). Indeed, f stretches each ray of the disk $\{(\theta, \rho) : \rho \leq 1\}$ to the ray of the square $\{(\theta, \rho)\colon \rho \leq g(\theta)\}$. Since g is continuous, we have that f and f^{-1} are also continuous which proves that f is a homeomorphism as we wanted.

Alternative solution (sketch): By rotating of angle $\pi/4$ and scaling with factor $1/\sqrt{2}$, we obtain that the square Q is homeomorphic to the l^1-ball $X \coloneqq \{(x,y) \in \mathbb{R}^2 : |x| + |y| \leq 1\}$. Then it is possible to check that X is homeomorphic to the unit disk B via the map $f : B \to X$ defined as $f(x,y) \coloneqq (\mathrm{sgn}(x)x^2, \mathrm{sgn}(y)y^2)$

Solution of 2.2:

(i) We need to show that

1. For every point $n \in \mathbb{Z}$, there is $B_{a,b} \in \mathcal{B}$ such that $n \in B_{a,b}$.
2. Given $B_{a,b}$, $B_{a',b'}$ and a number $n \in B_{a,b} \cap B_{a',b'}$, there is $B_{a'',b''}$ such that $n \in B_{a'',b''} \subseteq B_{a,b} \cap B_{a',b'}$.

For the first point, just notice that $B_{a,1} = \mathbb{Z}$, for every $a \in \mathbb{Z}$. For the second item, we claim that choosing $a'' = n$ and $b'' = bb'$ works. Indeed, $n \in B_{n,bb'}$, moreover we show that $B_{n,bb'} \subseteq B_{a,b}$ (the other inclusion is completely analogous). Since $n \in B_{a,b}$, there exists $k \in \mathbb{Z}$ such that $n = a + kb$. Thus, for every $h \in \mathbb{Z}$, we have $n + hbb' = a + kb + hbb' = a + (k + hb')b$, which is an element of $B_{a,b}$ as desired.

(ii) For each a, b with $b > 0$, we have that $B_{a,b}$ is open by definition. We need to show that is also closed, namely that it can be written as $\mathbb{Z} \setminus O$, where O is an open set. It is clear that $B_{a,b} \cup B_{a+1,b} \cup \dots \cup B_{a+(b-1),b} = \mathbb{Z}$. Moreover, we claim that for each $0 < r < b$, the sets $B_{a,b}$ and $B_{a+r,b}$ are disjoint. Indeed, suppose there were $k, h \in \mathbb{Z}$ such that $a + kb = a + r + hb$. Then we would have $r = (k - h)b$, which is impossible by choice of r. Therefore we obtain that

$$B_{a,b} = \mathbb{Z} \setminus \bigcup_{s=1}^{b-1} B_{a+s,b},$$

which proves that $B_{a,b}$ is closed since $\bigcup_{s=1}^{b-1} B_{a+s,b}$ is open.

(iii) If the set P was finite, then $\bigcup_{p\in P} B_{0,p}$ would be the finite union of closed sets, thus a closed set. Then, $\mathbb{Z} \setminus \bigcup_{p\in P} B_{0,p}$ would be open. However, by definition of prime numbers we have that $\mathbb{Z} \setminus \bigcup_{p\in P} B_{0,p} = \{1, -1\}$, which is not open. Thus P has to be infinite.

Solution of 2.3:

(i) We need to show that

1. For every $x \in \mathbb{R}$, there is $[a, b) \in \mathcal{B}$ such that $x \in [a, b)$.
2. Given $[a, b), [a', b') \in \mathcal{B}$ and $x \in [a, b) \cap [a', b')$, there exists $[a'', b'') \in \mathcal{B}$ such that $x \in [a'', b'') \subset [a, b) \cap [a', b')$.

The first point follows from the fact that, for every $x \in \mathbb{R}$, $x \in [\lfloor x \rfloor, x + 1) \in \mathcal{B}$, where $\lfloor x \rfloor$ is the integer part of x. For the second point, notice that either it holds $[a, b) \cap [a', b') = \emptyset$ or $[a, b) \cap [a', b') = [\max(a, a'), \min(b, b')) \in \mathcal{B}$, which implies what we want.

(ii) The interior of A is given by the union of elements of $\mathcal{B}$ contained in A, thus we have

$$\operatorname{int}(A) = \bigcup_{\substack{[a,b)\in\mathcal{B} \\ [a,b)\subset(1/2,2)}} [a, b) = \bigcup_{1<b<2} [1, b) = [1, 2).$$

On the other hand, the closure of A is the complement of the interior of $\mathbb{R} \setminus A$, which is given by

$$\operatorname{int}(\mathbb{R} \setminus A) = \bigcup_{\substack{[a,b)\in\mathcal{B} \\ [a,b)\cap(1/2,2)=\emptyset}} [a, b) = \bigcup_{\substack{a\in\mathbb{Z},\ a\le 0 \\ a<b<1/2}} [a, b) \cup \bigcup_{\substack{a\in\mathbb{Z},\ a\ge 2 \\ a<b}} [a, b) = (-\infty, 1/2) \cup [2, \infty),$$

which implies that $\overline{A} = [1/2, 2)$.

Solution of 2.4:

(i) Let O be an open subset of $X \times Y$. By definition of the product topology, $O = \bigcup_{i\in I}(U_i \times V_i)$ for some U_i open subsets of X and V_i open subsets of Y. Then

$$(f \times g)(O) = (f \times g)\left(\bigcup_{i\in I} U_i \times V_i\right) = \bigcup_{i\in I}(f \times g)(U_i \times V_i) = \bigcup_{i\in I} f(U_i) \times g(V_i).$$

Since the right hand side consists of a union of open sets (product of opens is open by definition of product topology), we get that $(f \times g)(O)$ is open.

(ii) Let $f, g\colon \mathbb{R} \to \mathbb{R}$ be the functions defined as $f(x) := x$, $g(x) := 0$ for all $x \in \mathbb{R}$. It is easy to see that f and g are closed function. We want to show that the function $(f \times g)\colon (x, y) \mapsto (x, 0)$ is not closed. Consider the closed set $C := \{(x, 1/x) \in \mathbb{R}^2 : x \in \mathbb{R} \setminus \{0\}\}$. Then $(f \times g)(C) = (\mathbb{R} \setminus \{0\}) \times \{0\}$, which is not a closed subset of $\mathbb{R}^2$. Thus we obtain that $f \times g$ is not closed.

To see that C is closed, consider the function $\eta\colon \mathbb{R}^2 \to \mathbb{R}$ defined as $\eta((x, y)) = xy$. Clearly, η is a continuous function. Moreover note that $C = \eta^{-1}(\{1\})$, which is closed since preimage of a closed set.

Solution of 2.5:

(i) First observe that $\mathrm{int}(A) \times \mathrm{int}(B) \subset A \times B$ and $\mathrm{int}(A) \times \mathrm{int}(B)$ is an open subset of $X \times Y$, by definition of product topology. Therefore $\mathrm{int}(A) \times \mathrm{int}(B) \subset \mathrm{int}(A \times B)$.

On the other hand, again by definition of product topology, we can write $\mathrm{int}(A \times B)$ as $\bigcup_{i\in I}(U_i \times V_i)$, where $U_i \subset X$ and $V_i \subset Y$ are open subsets of the respective spaces. However, since $U_i \times V_i \subset \mathrm{int}(A \times B) \subset A \times B$ for every $i \in I$, it holds that $U_i \subset A$ and $V_i \subset B$ for every $i \in I$. Hence, $U_i \subset \mathrm{int}(A)$ and $V_i \subset \mathrm{int}(B)$, because U_i and V_i are open. This proves the other inclusion since

$$\mathrm{int}(A \times B) = \bigcup_{i\in I}(U_i \times V_i) \subset \mathrm{int}(A) \times \mathrm{int}(B).$$

(ii) First note that $(X \times Y) \setminus (\overline{A} \times \overline{B}) = (\mathrm{int}(X \setminus A) \times Y) \cup (X \times \mathrm{int}(Y \setminus B))$ is open and therefore $\overline{A} \times \overline{B}$ is a closed subset of $X \times Y$ containing $A \times B$. As a result, $\overline{A \times B} \subseteq \overline{A} \times \overline{B}$, since $\overline{A \times B}$ is the minimal closed set containing $A \times B$.

For the other inclusion, let $(x, y) \in \overline{A} \times \overline{B}$, and let O be an open subset of $X \times Y$ such that $(x, y) \in O$. We want to show that $O \cap (A \times B) \neq \emptyset$, which implies that $(x, y) \subset \overline{A \times B}$. By definition of product topology, we have that $O = \bigcup_{i\in I}(U_i \times V_i)$, where $U_i \subset X$ and $V_i \subset Y$ are open sets for every $i \in I$. Since $(x, y) \in O$, there exists $i \in I$ such that $(x, y) \in U_i \times V_i$. Hence, using that $x \in \overline{A}$ and $y \in \overline{B}$, we obtain that $U_i \cap A \neq \emptyset$ and $V_i \cap B \neq \emptyset$, which proves that $O \cap (A \times B) \supseteq (U_i \times V_i) \cap (A \times B) \neq \emptyset$ as we wanted.

Solution of 2.6: Let $O \in \mathcal{T}_{Z,Y}$. Then $O = V \cap Z$ for some $V \in \mathcal{T}_Y$. However, by definition of subset topology on Y, we have in turn that $V = U \cap Y$ for some $U \in \mathcal{T}_X$. Thus, $O = (U \cap Y) \cap Z = U \cap Z$, because $Z \subseteq Y$. Hence we get $\mathcal{T}_{Z,Y} \subseteq \mathcal{T}_{Z,X}$.

For the other inclusion, let $O \in \mathcal{T}_{Z,X}$. Then $O = U \cap Z$ for some $U \in \mathcal{T}_X$. However, $V = U \cap Y \in \mathcal{T}_Y$, and thus $O = V \cap Z$ is contained also in $\mathcal{T}_{Z,Y}$.

Solution of 2.7: Given $x \in \mathrm{int}_X(A)$, there is an open set O of X that contains x and is contained in A. Note that $O \cap Y$ is an open set of Y that contains x and is contained in A. Thus $\mathrm{int}_X(A) \subseteq \mathrm{int}_Y(A)$.

The other inclusion does not hold in general. An example is given by choosing $Y \subset X$ not open and setting $A = Y$. Indeed, $A = \mathrm{int}_Y(A)$ (since $A = Y$ is open in Y), but $A \neq \mathrm{int}_X(A)$ (since $A = Y$ is not open in X). Concretely, let $X = \mathbb{R}$ and $A = Y = [0,1]$. Then $\mathrm{int}_Y(A) = [0,1] \neq (0,1) = \mathrm{int}_X(A)$.

Solution of 2.8: Let D be the set of the possible distances between different points of X, that is:

$$D := \{d(x,y) \,:\, x, y \in X, x \neq y\}.$$

Note that D is a finite set of positive numbers. In particular, D has a minimum $c > 0$. Then for every $x \in X$, the ball of radius $c/2$ and center x contains only x. Thus, for every $x \in X$, we have that $\{x\}$ is open in the distance topology, which is sufficient to prove that the distance topology coincides with the discrete topology.

Solution of 2.9: First note that $d_p(x,y) = d_p(y,x)$ for every $x, y \in \mathbb{Z}$ and that $d_p(x,x) = p^{-\max\{m \in \mathbb{N}\}} = 0$ for every $x \in \mathbb{Z}$. Now assume that $d_p(x,y) = 0$ for some $x, y \in \mathbb{Z}$. Then $p^m \mid x - y$ for every $m \in \mathbb{N}$ (indeed $p^{-m} > 0$ for every $m \in \mathbb{N}$), which implies $x = y$.

Hence, we only need to prove that the triangle inequality holds. Consider $x, y, z \in \mathbb{Z}$ and define $m_{x,y} := \max\{m \in \mathbb{N} \,:\, p^m \mid x - y\} = -\log_p(d_p(x,y))$, and analogously $m_{y,z}$ and $m_{x,z}$. Without loss of generality we can assume $d_p(x,z) \geq d_p(y,z)$, or equivalently $m_{x,z} \leq m_{y,z}$. Therefore $p^{m_{x,z}}$ divides both $x - z$ and $y - z$, thus it divides also $(x-z)-(y-z) = x-y$. This means that $m_{x,y} \geq m_{x,z}$, or equivalently $d_p(x,y) \leq d_p(x,z) = \max\{d_p(x,z), d_p(y,z)\} \leq d_p(x,z) + d_p(y,z)$.

Solution of 2.10: