3. Connectedness

3.1. Connected union of disks. Let $p = (-1, 0)$ and $q = (2, 0)$ be points in $\mathbb{R}^2$, and let $D_1 = \{z \in \mathbb{R}^2 : |z - p| < 1\}$ and $D_2 = \{z \in \mathbb{R}^2 : |z - q| < 2\}$. Which of the following subsets of $\mathbb{R}^2$ are connected?

(i) $D_1 \cup D_2$

(ii) $\overline{D_1} \cup D_2$

(iii) $\overline{D_1} \cup \overline{D_2}$

Note: You may use the fact that $\overline{D_1} = \{z \in \mathbb{R}^2 : |z - p| \leq 1\}$ *and* $\overline{D_2} = \{z \in \mathbb{R}^2 : |z - q| \leq 2\}$.

3.2. Connectedness in discrete topology. Let X be a set equipped with the discrete topology. Which subsets of X are connected?

3.3. Product arc connected. Let X and Y be non-empty topological spaces. Show that $X \times Y$ is path-connected if and only if both X, Y are path-connected.

3.4. From the circle to the real line. Let $S^1 = \{z \in \mathbb{R}^2 : |z - (0, 0)| = 1\}$ be the unit circle in $\mathbb{R}^2$, and let $f : S^1 \to \mathbb{R}$ be a continuous function. Show that there exists $z \in S^1$ such that $f(z) = f(-z)$. In particular, f is not injective.

Note: Given $z = (x_0, y_0) \in S^1$, *we denote by* $-z$ *the point* $(-x_0, -y_0) \in S^1$.

3.5. Closure and connectedness. Let X be a topological space and let A be a subset of X. Show that, if B is a subset of X with $A \subseteq B \subseteq \overline{A}$ and A is connected, then so is B.

3.6. Connectedness in the real line. Show that a subset of $\mathbb{R}$ is connected if and only if it is an interval.

3.7. Totally disconnected in the real line. Show that a subset of $\mathbb{R}$ is totally disconnected if and only if it does not contain any non-empty open interval.

Note: A topological space X is totally disconnected *if every connected subspace $A \subseteq X$ either is the empty set or consists of a single element.*

3.8. Complement of a countable set. If A is countable then $\mathbb{R}^2 \setminus A$ is path-connected.

3.9. Homeo(and diffeo)morphisms between Euclidean spaces. Show that $\mathbb{R}$ is not homeomorphic to $\mathbb{R}^2$ and that $\mathbb{R}^p$ is not diffeomorphic to $\mathbb{R}^q$ for any $p \neq q$.

Note: In fact it is true that $\mathbb{R}^p$ is not homeomorphic to $\mathbb{R}^q$ for any $p \neq q$ but the proof is much more involved. Later on in the course we will show the result in the case $p = 2$ and $q \geq 3$, while the proof of the general result needs finer algebraic topology tools.

3.10. Special subsets of the plane. Let G be a nonempty subset of $\mathbb{R}^2$ that is closed under addition, symmetric with respect to the origin and path-connected. Prove that G is a linear subspace of $\mathbb{R}^2$.

Note: The following theorem may be useful.

Jordan curve theorem. *Let $C \subseteq \mathbb{R}^2$ be a simple closed curve (i.e. C is the image of an injective continuous map $\gamma : S^1 \to \mathbb{R}^2$). Then $\mathbb{R}^2 \setminus C$ consists of exactly two connected components.*

3. Solutions

Solution of 3.1: Note that, by definition of the topology on $\mathbb{R}^2$, the sets D_1 and D_2 are open. It is easy to see that $D_1 \cap D_2 = \emptyset$. Indeed by triangular inequality, for each point $z \in \mathbb{R}^2$ we have $|z-p|+|z-q| \geq |p-q| = 3$. Therefore, if $|z-p| < 1$, then $|z-q| \geq 3 - |z-p| > 2$. This proves that if $z \in D_1$ then $z \notin D_2$ and the other case is analogous. Thus the set in (i) is the disjoint union of open sets, hence not connected.

We will now prove that the sets in (ii) and (iii) are connected, in particular that they are path-connected. Clearly for each point $z \in D_1$, there exists a path contained in D_1 that connects z to p (for example the segment connecting z and p). The same holds easily also for $\overline{D_1}$, D_2, $\overline{D_2}$. We claim that there exists a path that connects p and q that is contained in $\overline{D_1} \cup D_2 \subseteq \overline{D_1} \cup \overline{D_2}$, which implies that both of the unions above are path-connected.

Let $\gamma\colon [0,3] \to \mathbb{R}^2$ be the path defined as $\gamma(t) = (-1+t, 0)$. Clearly γ is a continuous path that connects p and q. We claim that the image of γ is contained in $\overline{D_1} \cup D_2$ (and hence in $\overline{D_1} \cup \overline{D_2}$). Indeed, for each $t \leq 1$, it holds $|\gamma(t)-p| = |(t-1,0)-(-1,0)| = |(t,0)| = t \leq 1$, hence $\gamma(t) \in \overline{D_1}$. Moreover, for each $1 < t \leq 3$, we have $|\gamma(t)-q| = |(t-1,0)-(2,0)| = |t-3| < 2$, thus $\gamma(t) \in D_2$.

Solution of 3.2: The only connected subsets are the empty set and the points of X, i.e. the subsets $\{x\}$ for $x \in X$. Since the set $\{x\}$ for $x \in X$ cannot be written as the disjoint union of *any* pair of non-empty sets, the points of X are trivially connected (and the same holds for the empty set). On the other hand, let $Y \subseteq X$ be a subset that contains at least two elements. Let $x \in Y$, the sets $\{x\}$ and $Y \setminus \{x\}$ are disjoint open sets (because in the discrete topology every set is open) and their union is Y. Hence, Y is not connected.

Solution of 3.3: We first prove that, if X and Y are path-connected, then $X \times Y$ is path-connected. In particular, for every $(x_0,y_0),(x_1,y_1) \in X \times Y$, we want to find a continuous curve $\gamma\colon [0,1] \to X \times Y$ such that $\gamma(0) = (x_0,y_0)$ and $\gamma(1) = (x_1,y_1)$. Since X and Y are both path-connected, there exist continuous curves $\gamma_X\colon [0,1] \to X$, $\gamma_Y\colon [0,1] \to Y$ such that $\gamma_X(0) = x_0$, $\gamma_X(1) = x_1$, $\gamma_Y(0) = y_0$ and $\gamma_Y(1) = y_1$. Then the curve $\gamma \coloneqq \gamma_X \times \gamma_Y$ satisfies the properties we want. Indeed the product of continuous functions is continuous, then γ is continuous. Moreover $\gamma(0) = (\gamma_X(0), \gamma_Y(0)) = (x_0,y_0)$ and $\gamma(1) = (\gamma_X(1), \gamma_Y(1)) = (x_1,y_1)$.

Viceversa, assume that $X \times Y$ is path-connected and denote by $\pi_X\colon X \times Y \to X$ and $\pi_Y\colon X \times Y \to Y$ the continuous projections of $X \times Y$ to X and Y, respectively. Then, since π_X and π_Y are surjective, we obtain that X and Y are path-connected (the image through a continuous map of a path-connected set is path-connected), which proves the other implication.

Solution of 3.4: To simplify the notation, let us parametrized S^1 by polar coordinates of angle $\theta \in \mathbb{R}$, with the usual convention that θ and θ' represent the same point if $\theta - \theta'$ is a multiple of 2π. Note that, if a point $z \in S^1$ is represented by angle θ in polar coordinates, then $-z \in S^1$ is represented by angle $\pi + \theta$.

Now consider a continuous function $f\colon S^1 \to \mathbb{R}$. If $f(0) = f(\pi)$, then we are done. Hence let us assume that $f(0)$ and $f(\pi)$ are different and consider the function $g\colon [0,\pi] \to \mathbb{R}$ defined as:

$$g(\theta) := f(\theta) - f(\pi + \theta).$$

It is clear that $g(0) = f(0) - f(\pi)$ and $g(\pi) = f(\pi) - f(0)$ have different sign, and that the function g is continuous. Thus, by the intermediate value theorem, there is $\theta \in [0,\pi]$ such that $g(\theta) = 0$, which concludes the proof.

Solution of 3.5: Suppose that B is not connected. Then there exist open sets O_1 and O_2 of X such that $B \subseteq O_1 \cup O_2$ and $B \cap O_1$, $B \cap O_2$ are non-empty and disjoint. Since $A \subseteq B \subseteq O_1 \cup O_2$, then one between $A \cap O_1$ and $A \cap O_2$ is empty, otherwise A would not be connected. So, without loss of generality, let us assume that $A \cap O_2 = \emptyset$, which implies that $A \subseteq O_1 \cap B$. Then consider the closed set $C := X \setminus O_2$. By construction $A \subseteq C$ and $(B \cap O_2) \cap C = \emptyset$, which implies $\overline{A} \setminus C \supseteq B \setminus C \supseteq B \cap O_2 \neq \emptyset$. However, this is a contradiction, since $\overline{A}$ is the minimal closed set that contains A.

Solution of 3.6: Obviously any interval of $\mathbb{R}$ is connected since it is path-connected. Viceversa, let C be a connected subset of $\mathbb{R}$ and assume by contradiction that C is not an interval. Then there exist $a, b \in C$ and $c \notin C$ such that $a < c < b$. Therefore, we have that $C = (C \cap (-\infty, c)) \cup (C \cap (c, \infty))$, which easily contradicts the connectedness of C, since $(-\infty, c)$ and (c, ∞) are open and disjoint subsets of $\mathbb{R}$.

Solution of 3.7: If a set contains a non-empty open interval, then clearly it is not totally disconnected. On the other hand, if a set C is not totally disconnected, then there is a connected subset A of C that has more than one point. By the previous exercise, A is an interval with at least two points. Observe that $\mathrm{int}(A)$ is itself an (open) interval with at least two points, which concludes the proof.

Solution of 3.8: Consider two points $x, y \in \mathbb{R}^2 \setminus A$. Denote by $m := (x+y)/2$ the middle point between x and y and by v a vector orthogonal to $x - y$. For every $s \in [0,1]$ consider the continuous curve $\gamma^s\colon [0,1] \to \mathbb{R}^2$ defined as the union of the segment connecting x to $m + sv$ and the segment connecting $m + sv$ to y. Explicitly, γ^s is defined as

$$\gamma^s(t) := \begin{cases} (1-2t)x + 2t(m+sv) & \text{for } 0 \le t \le 1/2 \\ 2(1-t)(m+sv) + (2t-1)y & \text{for } 1/2 \le t \le 1. \end{cases}$$

Observe that, for every $s \neq s'$, the images of γ^s and $\gamma^{s'}$ intersect only at $x, y \notin A$. Therefore there exists $\bar{s} \in [0,1]$ such that $\gamma^{\bar{s}}([0,1]) \cap A = \emptyset$, since s can take values in the uncountable set $[0,1]$ while A is countable. Hence we have that $\gamma^{\bar{s}}$ is a continuous curve $\gamma^{\bar{s}}\colon [0,1] \to \mathbb{R}^2 \setminus A$ connecting x and y, which proves that $\mathbb{R}^2 \setminus A$ is path-connected by arbitrariness of $x, y \in \mathbb{R}^2 \setminus A$.

Solution of 3.9: To prove that $\mathbb{R}$ and $\mathbb{R}^2$ are not homeomorphic, it is sufficient to observe that $\mathbb{R} \setminus \{p\}$ is disconnected for any $p \in \mathbb{R}$ while $\mathbb{R}^2 \setminus \{q\}$ is connected for any $q \in \mathbb{R}^2$.

Hence, now let us show that $\mathbb{R}^p$ and $\mathbb{R}^q$ are not diffeomorphic for any $p \neq q$. Assume by contradiction that $\mathbb{R}^p$ and $\mathbb{R}^q$ are diffeomorphic for some $p > q$ and consider a diffeomorphism $f : \mathbb{R}^p \to \mathbb{R}^q$ with inverse $f^{-1} : \mathbb{R}^q \to \mathbb{R}^p$.

Let us denote by A, B the matrices representing the differentials $D_0 f, D_{f(0)} f^{-1}$ of f at 0 and of f^{-1} at $f(0)$, respectively. Then, since $f^{-1} \circ f = \mathrm{id}_{\mathbb{R}^p}$, we have that

$$\mathrm{id}_{\mathbb{R}^p} = D_0(\mathrm{id}_{\mathbb{R}^p}) = D_0(f^{-1} \circ f) = D_{f(0)} f^{-1} \circ D_0 f = BA.$$

However, this is a contradiction since the rank of BA is at most $q < p$, because it is a product of a $p \times q$ matrix with a $q \times p$ matrix.

Solution of 3.10: