4. Compactness

Chef's table

Starting this week, based on some requests I have received, I would like to add a few comments about the problem set, so to guide those among you who wish to focus on just five or six of them. As a general remark, even if you decide to write down accurate solutions only to a certain subset of exercises, it may be a great idea to take some time to think about all of them (with the exception of the challenge problem, if you have time constraints). Meditating on a problem without a piece of paper and a pen in front you is very good (and helpful) practice, which reinforces your abstraction skills and gets you closer to a research-type experience. It may be difficult at first, but you should give it a try.

That said, I envision two types of tasting menus. A *lighter option* might be **4.1 - 4.2 - 4.3 - 4.4 - 4.5 - 4.6**: these exercises are all pretty short, but they provide a good training. In particular, Problem 4.5 is very instructive. A *more demanding option* could be **4.5 - 4.7 - 4.8 - 4.9**: the combination of 4.8 and 4.9 provides a complete proof for a (very important!) characterisation of compact sets in Euclidean spaces, and the statement should be known to all students in the class. Overall, this couple of problems is rather lengthy to be written down in detail, but provides an excellent practice both from the technical viewpoint and from a *writing in the Major* perspective. Finally, let me note that Problem 4.6 is a very basic but helpful result in Real Analysis: for instance, it makes it much simpler to prove that the set of limit points of the sequence $a_n = \sin(n)$ coincides with the closed interval $[-1, 1]$, which would be a lot harder to prove with purely elementary tools.

4.1. Discrete topology ✍. Let X be a set equipped with the discrete topology. Characterize the compact subspaces of X.

4.2. Finite intersection property ✍. We say that a family $\mathcal{A}$ of subsets of a topological space X has the *finite intersection property* if for each (non-empty) finite subfamily $\mathcal{F}$ of $\mathcal{A}$ we have that

$$\bigcap_{A\in\mathcal{F}} A \neq \emptyset.$$

Show that a topological space X is compact if and only if, for every family of closed subsets $\mathcal{A}$ that has the finite intersection property, we have that

$$\bigcap_{A\in\mathcal{A}} A \neq \emptyset.$$

4.3. Intersection of compact sets ✍. Let X be a compact topological space, O be an open subset of X an $\{C_i\}_{i\in I}$ be a (possibly infinite) family of closed sets such that

$$\bigcap_{i\in I} C_i \subseteq O.$$

Show that it is possible to find a *finite* set of indices $\{i_1, \ldots, i_n\} \subseteq I$ such that

$$\bigcap_{k=1}^{n} C_{i_k} \subseteq O.$$

4.4. Finite number of digits ⚙. Given a topological space X, let X' be the subspace of X obtained by removing all the isolated points of X, i.e. all the points of X which are open and closed in X. Let B_n be the subspace of $[0, 1]$ that consists of all the numbers having a base 2 decimal expansion $0.a_1a_2a_3\ldots$ in which at most n of the digits a_i are 1, and let $B := \cup_{n\in\mathbb{N}} B_n$. Determine B' and B'_n for every $n \in \mathbb{N}$. Deduce that there for each $n \in \mathbb{N}$ there is a space X such that the sequence

$$X \supseteq X' \supseteq X'' \supseteq \ldots$$

becomes the empty set after exactly n stages.

4.5. The cofinite topology ✎. Let $\mathcal{T}$ be the family of subsets of the real line $\mathbb{R}$ defined as

$$\mathcal{T} := \emptyset \cup \{\mathbb{R} \setminus F \,:\, F \subset \mathbb{R} \text{ is finite}\}.$$

(i) Check that $\mathcal{T}$ is a topology and that $(\mathbb{R}, \mathcal{T})$ is compact.

(ii) Let $\mathcal{T}_{\text{std}}$ be the standard topology on $\mathbb{R}$. Show that $(\mathbb{R}, \mathcal{T})$ and $(\mathbb{R}, \mathcal{T}_{\text{std}})$ are not homeomorphic.

4.6. Limit points of a sequence ⚙. Let $\{x_n\}_{n\in\mathbb{N}} \subseteq \mathbb{R}$ be a sequence of points in $\mathbb{R}$. We say that $y \in \mathbb{R}$ is a *limit point* for this sequence if there exists a subsequence $\{x_{n_k}\}_{k\in\mathbb{N}}$ of $\{x_n\}_{n\in\mathbb{N}}$ that converges to y. Prove that the set of the limit points of any sequence in $\mathbb{R}$ is closed.

4.7. Neighborhood of a set ⚙. Let C be a closed subset of $\mathbb{R}^n$ and let A be an open subset of $\mathbb{R}^n$ that contains C. For every $\varepsilon > 0$, define $C_\varepsilon := \{x \in \mathbb{R}^n \,:\, d(x, C) < \varepsilon\}$. Prove that, if C is compact, then there exists $\varepsilon > 0$ such that $C_\varepsilon \subseteq A$. Is the conclusion true removing the hypothesis of C being compact?

Note: Given any $x \in \mathbb{R}^n$ and $S \subseteq \mathbb{R}^n$, the distance of x to S is defined as $d(x, S) := \inf_{s\in S}|x - s|$.

4.8. Preparation to Problem 4.9 ⚙. Before facing Problem 4.9, we need the following preliminary facts.

(i) Let (X, d) be a metric space and assume that X is *separable*, which means that X contains a countable dense subset. Prove that any open cover $\mathcal{O}$ of X admits a countable subcover.

(ii) Let (X, d) be a complete metric space. Then a subset $Y \subseteq X$ is closed if and only if it is complete. Observe that this applies in particular to $X = \mathbb{R}^n$ with the Euclidean distance.

(iii) Prove that a subset of $\mathbb{R}^n$ is totally bounded if and only if it is bounded. Show that this is not true in general in a complete metric space (X, d).

4.9. Equivalent notions of compactness. Given a metric space (X, d), show that the following conditions are equivalent:

(C1) The space X is compact (i.e. every open cover of X admits a finite subcover).

(C2) The space X is sequentially compact (i.e. every sequence $\{x_n\}_{n\in\mathbb{N}} \subseteq X$ admits a converging subsequence).

(C3) The space X is complete (i.e. every Cauchy sequence $\{x_n\}_{n\in\mathbb{N}}$ converges to some $x \in X$) and totally bounded (i.e. for every $\varepsilon > 0$ there exists a finite set of points $x_1, \ldots, x_k \in X$ such that $X \subseteq \cup_{i=1}^{k} B(x_i, \varepsilon)$).

Note: Observe that, thanks to (ii) *and* (iii) *in Problem 4.8, this statement is equivalent to the analogous one seen in class in the case when* (X, d) *is a subspace of a Euclidean space.*

4.10. Union of strictly convex compact sets. Prove that it is not possible to obtain $\mathbb{R}^n$ as a countable union of strictly convex compact sets that are pairwise disjoint.

4. Solutions

Solution of 4.1: We want to prove that a subspace $A \subseteq X$ is compact if and only if is finite. Obviously every finite subset of X is compact, hence let us prove the other implication. Consider an infinite subspace $A \subseteq X$. Then $\{x\}$ is open in A for every $x \in A$, since A inherits the discrete topology from X. Therefore $\mathcal{O} := \{\{x\} : x \in A\}$ is an infinite open covering of A. However, patently, $\mathcal{O}$ does not admit any finite subcover, which proves that A is not compact.

Solution of 4.2: Assume that X is compact, and suppose that there exists a family $\mathcal{A}$ of closed subsets of X such that the intersection of all the elements of $\mathcal{A}$ is empty, but any finite subcollection of $\mathcal{A}$ has non-empty intersection (i.e. $\mathcal{A}$ has the finite intersection property). For a set $A \subseteq X$, we denote by A^{c} the complement of A, that is $A^{\mathsf{c}} := X \setminus A$. Since the intersection of all the elements $A \in \mathcal{A}$ is empty, we have that

$$\bigcup_{A \in \mathcal{A}} A^{\mathsf{c}} = X.$$

In particular, $\mathcal{A}^{\mathsf{c}} = \{A^{\mathsf{c}} : A \in \mathcal{A}\}$ is an open cover of X. Thus, there exists a finite subfamily of $\mathcal{A}^{\mathsf{c}}$ that still covers X, which means that there exists a finite subfamily $\mathcal{F}$ of $\mathcal{A}$ such that $\{A^{\mathsf{c}} : A \in \mathcal{F}\}$ covers X. However, this implies that

$$\bigcap_{A \in \mathcal{F}} A = \emptyset,$$

which is a contradiction.

On the other hand, assume that X is not compact. Then there exists an infinite family of open sets $\mathcal{B}$ that covers X such that for every finite subfamily $\mathcal{G}$ of $\mathcal{B}$, the union of the elements of $\mathcal{G}$ does not cover X. Using the same argument as above, we see that $\mathcal{A} := \mathcal{B}^{\mathsf{c}}$ has the finite intersection property, but the intersection of all its elements is the empty set, which proves the other implication.

Solution of 4.3: Note that $O^{\mathsf{c}} = X \setminus O$ is a closed subset of X, thus it is compact. Moreover the family $\{C_i^{\mathsf{c}}\}_{i \in I}$ is an open cover of O^{c}, and thus there exists a finite subfamily $\mathcal{F} = \{C_{i_1}^{\mathsf{c}}, \dots, C_{i_n}^{\mathsf{c}}\}$ that covers O^{c}. In particular,

$$\bigcup_{k=1}^{n} C_{i_k}^{\mathsf{c}} \supseteq O^{\mathsf{c}} \quad \Longrightarrow \quad \bigcap_{k=1}^{n} C_{i_k} \subseteq O.$$

Solution of 4.4: We start by proving that, for every $x \in B_{n-1}$ with $n \geq 0$, there exists a sequence $\{x_k\}_{k>0} \subseteq B_n \setminus \{x\}$ that converges to x. By definition, there exist $0 < i_1 < \dots < i_m$, with $m \leq n-1$, such that $x = 2^{-i_1} + \ldots + 2^{-i_m}$. Then, for all $k > 0$, define

$$x_k := x + 2^{-(i_m+k)} = 2^{-i_1} + \ldots + 2^{-i_m} + 2^{-(i_m+k)}.$$

Observe that the base 2 decimal representation of x_k has exactly one 1 more than x, therefore $x_k \in B_n$ for all $k > 0$. Moreover $|x - x_k| = 2^{-(i_m+k)} \to 0$, which proves that $\{x_k\}_{k>0}$ converges to x as we wanted.

As a result, the set $B = \cup_{n\in\mathbb{N}} B_n$ does not have isolated points, and thus $B' = B$. We now want to show that $B'_n = B_{n-1}$. We have already proved that any point of B_{n-1} is not isolated in B_n, therefore $B_{n-1} \subseteq B'_n$. Therefore we just need to show that every $x \in B_n \setminus B_{n-1}$ is isolated in B_n. Write $x = 2^{-i_1} + \ldots + 2^{-i_n}$ with $0 < i_1 < \ldots < i_n$ as before and consider $y \in B_n \setminus \{x\}$. Since $y \neq x$ and y has at most n digits in its base 2 representation, there exists $k = 1, \ldots, n$ such that y does not have the i_kth digit in its base 2 representation. Hence, we obtain that $|x - y| \geq 2^{-i_n}$, which proves that x is isolated in B_n.

Solution of 4.5:

(i) Let us check that $\mathcal{T}$ is a topology:

- $\emptyset$ and $\mathbb{R}$ are trivially contained in $\mathcal{T}$.
- If $\mathbb{R} \setminus F_1$ and $\mathbb{R} \setminus F_2$ are contained in $\mathcal{T}$ (i.e. F_1 and F_2 are finite), then their intersection $(\mathbb{R} \setminus F_1) \cap (\mathbb{R} \setminus F_2) = \mathbb{R} \setminus (F_1 \cup F_2)$ is contained in $\mathcal{T}$ since $F_1 \cup F_2$ is finite. Thus $\mathcal{T}$ is closed under finite intersections.
- If $\{\mathbb{R} \setminus F_i\}_{i\in I}$ is a family of sets in $\mathcal{T}$, then their union $\bigcup_{i\in I}(\mathbb{R} \setminus F_i) = \mathbb{R} \setminus \cap_{i\in I} F_i$ is contained in $\mathcal{F}$, since the intersection of (arbitrarily many) finite sets is finite. Hence we have that $\mathcal{T}$ is closed under union.

Let us now prove that $(\mathbb{R}, \mathcal{T})$ is compact. Consider an open cover $\mathcal{O} = \{\mathbb{R} \setminus F_i\}_{i\in I}$ of $\mathbb{R}$. This means that $\bigcup_{i\in I}(\mathbb{R}\setminus F_i) = \mathbb{R}$, or equivalently $\bigcap_{i\in I} F_i = \emptyset$. Consider $F_1 = \{x_1, \ldots, x_n\}$. Then, for all $k = 1, \ldots, n$, there exists $i_k \in I$ such that $x_k \notin F_{i_k}$, since the intersection of all F_i's is empty. Therefore $\bigcap_{k=1}^n F_{i_k} \cap F_1 = \emptyset$, which prove that $\mathcal{O}' := \{\mathbb{R} \setminus F_1\} \cup \{\mathbb{R} \setminus F_{i_k}\}_{k=1}^n$ is finite subcover of $\mathcal{O}$.

(ii) The two topological spaces are not homeomorphic since $(\mathbb{R}, \mathcal{T}_{\text{std}})$ is not compact. To prove this, consider the following open cover of $\mathbb{R}$ with respect to $\mathcal{T}_{\text{std}}$: for each $m \in \mathbb{Z}$, let $O_m = (m-1, m+1)$, and let $\mathcal{O} = \{O_m : m \in \mathbb{Z}\}$. Observe that the only element of $\mathcal{O}$ that contains an integer $m \in \mathbb{Z}$ is O_m. Therefore, removing any element from $\mathcal{O}$, we do not have a cover anymore. This shows that $(\mathbb{R}, \mathcal{T}_{\text{std}})$ is not compact as we wanted.

Solution of 4.6: Consider the set L of limit points of a sequence $\{x_n\}_{n\in\mathbb{N}}$ and let $\bar{x} \in \overline{L}$. By the definition of closure, for every open set $U \subseteq \mathbb{R}$ that contains $\bar{x}$, we can find $y \in L \cap U$. In particular, for every $k \in \mathbb{N}$, there exists $y_k \in L \cap (\bar{x} - 1/k, \bar{x} + 1/k)$. Since y_k is a limit point for the sequence and $(\bar{x} - 1/k, \bar{x} + 1/k)$ is an open neighborhood of y_k, then there exists $n_k \in \mathbb{N}$ such that $x_{n_k} \in (\bar{x} - 1/k, \bar{x} + 1/k)$. Observe that the subsequence $\{x_{n_k}\}_{k\in\mathbb{N}}$ of $\{x_n\}_{n\in\mathbb{N}}$ converges to $\bar{x}$ as $k \to \infty$ (it is not difficult to see that we can also choose n_k non-decreasing in $k \in \mathbb{N}$). This means that $\bar{x}$ is a limit point for the sequence and thus $\bar{x} \in L$, which concludes the proof that $\overline{L} = L$.

Solution of 4.7: Consider the continuous function $f : C \to [0, \infty)$ defined as $f(x) := d(x, \mathbb{R}^n \setminus A)$ (check the continuity!). Assume that $f(x) = 0$ for some $x \in C$, then there exists a sequence $\{x_k\}_{k\in\mathbb{N}} \subseteq \mathbb{R}^n \setminus A$ such that $|x - x_k| \to 0$. However, since $\mathbb{R}^n \setminus A$ is closed, this implies that $x \in \mathbb{R}^n \setminus A$, which contradicts the fact that $x \in C \subseteq A$. Therefore $f(x) > 0$ for every $x \in C$.

Now observe that f is a continuous function defined on a compact set C, hence there exists a point $x_0 \in C$ that achieves the minimum of f, namely $f(x_0) = \inf_{x\in C} f(x)$. We want to prove that $C_\varepsilon \subseteq A$ for $\varepsilon := f(x_0)$. By definition of C_ε, this is equivalent to proving that $d(y, C) \geq \varepsilon$ for every $y \in \mathbb{R}^n \setminus A$. However, this follows directly from the fact that $\varepsilon = f(x_0) = \inf_{x\in C} d(x, \mathbb{R}^n \setminus A) = \inf_{x\in C} \inf_{y\in\mathbb{R}^n\setminus A} |x - y| = \inf_{y\in\mathbb{R}^n\setminus A} d(y, C)$.

Finally, we want to prove that, if C is not compact, then the result does not hold. Let us define the closed set $C := \{(x, 1/x) \in \mathbb{R}^2 \,:\, x > 0\}$ and consider $A := \{x, y \in \mathbb{R}^2 \,:\, y > 0\}$, which is an open set containing C. Given any $\varepsilon > 0$, note that $|(2/\varepsilon, 0) - (2/\varepsilon, \varepsilon/2)| < \varepsilon$; hence $(2/\varepsilon, 0) \in C_\varepsilon$, but $(2/\varepsilon, 0) \notin A$, which proves that $C_\varepsilon \not\subseteq A$.

Solution of 4.8:

(i) First observe that X admits a countable basis for the topology generated by the metric d. Indeed, denoting by D a countable dense subset of X, the countable set

$$\mathcal{B} := \{B(x, q) \,:\, x \in D, q \in \mathbb{Q}_{>0}\}$$

is a basis for the topology of metric space on X. Indeed consider any open set $U \subseteq X$ and any point $x_0 \in U$. By definition of metric topology, there exists $y \in U$ and $r > 0$ such that $x_0 \in B(y, r) \subseteq U$. Then observe that $B(x_0, r') \subseteq B(y, r) \subseteq U$ with $r' := r - d(x_0, y) > 0$, by the triangle inequality. Since D is dense, there exists $x \in D \cap B(x_0, r'/2)$, therefore we have that $x_0 \in B(x, r'/2) \subseteq B(x_0, r') \subseteq U$, again by triangle inequality. Now take $q \in \mathbb{Q}_{>0}$ such that $d(x_0, x) < q < r'/2$ (which is possible since $d(x_0, x) < r'/2$), then $x_0 \in B(x, q) \subseteq B(x, r'/2) \subseteq U$. Note that $B(x, q) \in \mathcal{B}$, thus we proved that $\mathcal{B}$ is a basis for the topology.

Note: Here we proved that any separable metric space is second-countable (see Problem set 5 for the definition). However, this is not true in general. In fact there exist separable first-countable topological spaces (see again Problem set 5 for the definition) that are not second-countable.

Now we want to prove that, if a topological space X has a countable basis for its topology, then every open cover admits a countable subcover. Denote by $\mathcal{B} = \{B_n\}_{n\in\mathbb{N}}$ a countable basis of the topology and consider an open cover $\mathcal{O}$. Define $I \subset \mathbb{N}$ as the set of indices $n \in \mathbb{N}$ such that there exists $O_n \in \mathcal{O}$ containing B_n. Then define $\mathcal{O}' := \{O_n \,:\, n \in I\}$, where for every $n \in I$ we make a choice of $O_n \in \mathcal{O}$ such that $B_n \subseteq O_n$. We claim that $\mathcal{O}'$ is a countable subcover of $\mathcal{O}$. The fact that $\mathcal{O}'$ is countable is obvious, hence let us prove that it is a cover. Consider $x \in X$, then there exists $O \in \mathcal{O}$ such that $x \in O$. Since $\mathcal{B}$ is a basis for the topology, we can pick $B_n \in \mathcal{B}$ such that $x \in B_n \subseteq O$. In particular $n \in I$, hence $x \in B_n \subseteq O_n$ for some $O_n \in \mathcal{O}'$, which proves that $\mathcal{O}'$ is a cover.

(ii) First assume that $Y \subseteq X$ is closed and consider a Cauchy sequence $\{x_n\}_{n\in\mathbb{N}} \subseteq Y$, which converges to some point $x \in X$ by completeness of X. Hence, since Y is closed and $x_n \in Y$ for every $n \in \mathbb{N}$, x is contained in Y too, which proves that Y is complete.

Viceversa, assume that $Y \subseteq X$ is complete and consider a sequence $\{x_n\}_{n\in\mathbb{N}} \subseteq Y$ converging to some point $x \in X$. Then, for every $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that $d(x_n, x) \leq \varepsilon/2$ for every $n \geq N$. As a result we obtain that $d(x_n, x_m) \leq d(x_n, x) + d(x_m, x) \leq \varepsilon$ for every $n, m \geq N$, which proves that $\{x_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence. Therefore this sequence converges to some point $y \in Y$ by completeness of Y. However, notice that y must coincide with x since the limit of a sequence in a metric space is unique. Hence we have shown that $x \in Y$, so Y is closed.

(iii) Consider a subset Y of $\mathbb{R}^n$. Obviously if Y is totally bounded then it is bounded. Indeed, there exist $x_1, \dots, x_k \subseteq Y$ such that $Y \subseteq \cup_{i=1}^k B(x_i, 1)$. Therefore for every $x, y \in Y$ we have that $|x - y| \leq 2 + \max_{i,j=1\dots,k} |x_i - x_j| < \infty$.

For the other implication, first observe that a subset Y of a totally bounded space (X, d) is totally bounded. Indeed, given any $\varepsilon > 0$, there exist $x_1, \dots, x_k \in X$ such that $X \subseteq \cup_{i=1}^k B(x_i, \varepsilon/2)$. Then, for every $i = 1, \dots, k$, choose $y_i \in Y \cap B(x_i, \varepsilon/2)$ (if it exists, otherwise we just ignore the index). We claim that $Y \subseteq \cup_{i=1}^k B(y_i, \varepsilon)$. This follows from the fact that $B(x_i, \varepsilon/2) \subseteq B(y_i, \varepsilon)$ (you can check it, using the triangle inequality).

Given this preliminary fact, we can now prove easily that if $Y \subseteq \mathbb{R}^n$ is bounded then it is totally bounded. Indeed, by boundedness of Y, there exists $R > 0$ such that $Y \subseteq [-R, R]^n$ and we will now show that $[-R, R]^n \subseteq \mathbb{R}^n$ is totally bounded. Taken any $\varepsilon > 0$, we cover $[-R, R]^n$ with a finite number of cubes $C_1, \dots, C_k$ with edges of length less that $2\varepsilon/\sqrt{n}$ (this is easily obtained by covering the interval $[-R, R]$ with a finite number of intervals of length less than $2\varepsilon/\sqrt{n}$ and then considering the "product cover"). Then denote by $x_1, \dots, x_k$ the center of the cubes and observe that $C_i \subseteq B(x_i, \varepsilon)$ for every $i = 1, \dots, k$, since the diameter of C_i is $\sqrt{n} \cdot 2\varepsilon/\sqrt{n}$. Therefore $[-R, R]^n \subseteq \cup_{i=1}^k C_i \subseteq \cup_{i=1}^k B(x_i, \varepsilon)$, which proves the total boundedness of $[-R, R]^n$ by arbitrariness of $\varepsilon > 0$.

Solution of 4.9: We will prove that (C1) is equivalent to (C2), which is in turn equivalent to (C3).

(C1) $\Longrightarrow$ (C2) Assume by contradiction that X is compact but not sequentially compact. In particular there exists a sequence $\{x_n\}_{n\in\mathbb{N}}$ without converging subsequences. Then define

$$\mathcal{O} := \{O \subseteq X \,:\, O \text{ open}, O \text{ contains a finite number of elements in } \{x_n\}_{n\in\mathbb{N}}\}.$$

Observe that $\mathcal{O}$ is an open cover of X. Indeed, for every $x \in X$, there exists an open neighborhood O of x that does not contain elements of $\{x_n\}_{n\in\mathbb{N}}$ eventually in $n \in \mathbb{N}$. Otherwise x would be an accumulation point for $\{x_n\}_{n\in\mathbb{N}}$ and thus it would exists a subsequence of $\{x_n\}_{n\in\mathbb{N}}$ converging to x, since any metric space is first countable (this is the only step in this implication where we use the hypothesis of metric space and in fact first countable would be sufficient). Hence, by compactness of X, there exists a finite subcover $\mathcal{O}'$ of $\mathcal{O}$. Since $\mathcal{O}'$ is a finite cover of X and $\{x_n\}_{n\in\mathbb{N}}$ is an infinite sequence, there exists $O' \in \mathcal{O}'$ that contains an infinite number of elements of $\{x_n\}_{n\in\mathbb{N}}$. However, this contradicts the fact that $O' \in \mathcal{O}' \subseteq \mathcal{O}$.

(C2) $\implies$ (C1) Let us assume that X is sequentially compact, we want to prove that it is compact. We first show that X is separable. First observe that X is bounded, otherwise it is easy to construct a sequence without converging subsequences ("going to infinity"). We construct the following sequence: we fix some $x_0 \in X$ and then we define x_{n+1} for $n \geq 0$ in such a way that

$$\min_{i=1,\dots,n} d(x_{n+1}, x_i) \geq \frac{1}{2} \sup_{x\in X} \min_{i=1,\dots,n} d(x, x_i). \tag{1}$$

Observe that the term on the right hand side is finite by boundedness of X, thus it is possible to find x_{n+1} as required. We want to show that $\{x_n\}_{n\in\mathbb{N}}$ is a dense subset of X. Since X is sequentially compact, $\{x_n\}_{n\in\mathbb{N}}$ admits a converging subsequence $\{x_{n_m}\}_{m\in\mathbb{N}}$. This implies that $\min_{i=1,\dots,n_m-1} d(x_{n_m}, x_i) \leq d(x_{n_m}, x_{n_{m-1}})$ converges to 0 as $m \to \infty$. As a result $\sup_{x\in X} \min_{i=1,\dots,n_m} d(x, x_i) \to 0$ as $m \to \infty$, by (1). However, from this it follows directly that

$$\sup_{x\in X} \min_{i=1,\dots,n} d(x, x_i) \to 0 \quad \text{as } n \to \infty,$$

which proves that $\{x_n\}_{n\in\mathbb{N}}$ is a (countable) dense subset of X.

Now consider an open cover $\mathcal{O}$ of X. Since X is separable, we can extract a countable subcover $\mathcal{O}' = \{O_k\}_{k\in\mathbb{N}}$ of $\mathcal{O}$ by (i) in Problem 4.8. Assume by contradiction that $\mathcal{O}'$ does not admit any finite subcover, then $\cup_{k=1}^n O_k \neq X$ for any $n \in \mathbb{N}$. In particular there exists $x_n \in X \setminus \cup_{k=1}^n O_k$ for any $n \in \mathbb{N}$. Since X is sequentially compact, the sequence $\{x_n\}_{n\in\mathbb{N}}$ has a subsequence converging to some point $x \in X$. However observe that $x \in X \setminus \cup_{k=1}^n O_k$ for all $n \in \mathbb{N}$, since the sequence $\{x_n\}_{n\in\mathbb{N}}$ is eventually contained in $X \setminus \cup_{k=1}^n O_k$, which is closed. Therefore, $x \in X \setminus \cup_{n\in\mathbb{N}} O_n$, which is a contradiction since $X \setminus \cup_{n\in\mathbb{N}} O_n = \emptyset$.

(C2) $\implies$ (C3) Let us assume that X is sequentially compact. Given a Cauchy sequence $\{x_n\}_{n\in\mathbb{N}}$, there exists a subsequence converging to some $x \in X$. However, it is not difficult to check (do it!) that if a subsequence of a Cauchy sequence converges to some point $x \in X$, then the whole sequence converges to such a point. Therefore X is complete The proof that X is totally bounded is analogous to the proof X is separable in the previous implication ((C2) $\implies$ (C1)), hence we leave it for the reader.

(C3) $\implies$ (C2) Let us assume that X is complete and totally bounded and consider a sequence $\{x_n\}_{n\in\mathbb{N}}$. We want to prove that this sequence admits a converging subsequence. Since X is totally bounded, for every $m \in \mathbb{N}$ there exists a finite cover $\mathcal{O}_m$ of metric balls of radius $1/m$. Note that there exists a subsequence $\{x_n^1\}_{n\in\mathbb{N}}$ of $\{x_n\}_{n\in\mathbb{N}}$ such that all its elements are contained in the same $O_1 \in \mathcal{O}_1$ (this follows from the fact that $\mathcal{O}_1$ is a finite cover of X). Analogously, for every $m > 1$, we can find a subsequence $\{x_n^m\}_{n\in\mathbb{N}}$ of $\{x_n^{m-1}\}_{n\in\mathbb{N}}$ such that all its elements are contained in the same $O_m \in \mathcal{O}_m$. Finally, with a diagonal argument, we consider the sequence $\{x_k^k\}_{k\in\mathbb{N}}$ (which is a subsequence of $\{x_n\}_{n\in\mathbb{N}}$). Observe that $\{x_k^k\}_{k\in\mathbb{N}}$ is eventually contained in $O_m \in \mathcal{O}_m$ for every $m \in \mathbb{N}$. Hence in particular $\{x_k^k\}_{k\in\mathbb{N}}$ is a Cauchy sequence, because O_m is a metric ball of radius $1/m$.

Therefore $\{x_k^k\}_{k\in\mathbb{N}}$ converges to some point $x \in X$ by completeness of X. This proves that $\{x_n\}_{n\in\mathbb{N}}$ has a converging subsequence and thus that X is sequentially compact.

Solution of 4.10: