5. Separation axioms and related stories

Chef's table

This week, the problem set we present is a lot shorter than usual (our solutions are less than two pages altogether). If you set aside the challenge problem, the other nine exercises could be defined as *quick review questions* and provide a good idea of the sort of tasks you will face in Section I.b of the exam. So you may all want to go for the complete tasting menu this week.

Among these problems, 5.7-5.8-5.9 concern separable spaces (namely: topological spaces containing a countable dense subset). The first two exercises are key facts that are often employed in Functional Analysis (stay tuned...). Instead, Problem 5.9 indicates what is perhaps the simplest example of a non-separable metric (in fact: Banach) space, and all students should have it in mind.

Lastly, the challenge problem this week (which is perhaps a bit more accessible than usual) shows that the equivalence we proved in Problem 4.9 last week does not hold as we move outside of the kingdom of metric spaces.

Remind: A topological space X is said to be *first-countable* if each point has a countable basis of neighborhoods, i.e. for every point $x \in X$ there exists a countable family $\mathcal{B}$ of open neighborhoods of x such that for every open set U that contains x there is $B \in \mathcal{B}$ with $B \subseteq U$. On the other hand, we say that a topological space is *second-countable* if it admits a countable basis for its topology.

5.1. Hausdorff and infinite products ✍. Let $\{X_i\}_{i\in I}$ be a family of Hausdorff topological spaces. Show that $X = \prod_{i\in I} X_i$ is a Hausdorff space.

5.2. Example of non-Hausdorff space ✍. Give an example of a topological space that is not Hausdorff.

5.3. Continuous bijection and Hausdorff ✍. Let X and Y be topological spaces. Suppose that Y is Hausdorff and that there is a continuous bijection $f\colon X \to Y$. Show that X is Hausdorff.

5.4. Subsets of Hausdorff spaces are Hausdorff ✍. Let X be a Hausdorff topological space and let Y be a subset of X. Show that Y is Hausdorff with respect to the induced topology.

5.5. Convergence and basis ⚙. Let X be a first-countable topological space, x be a point of X and $\{O_k\}_{k\in\mathbb{N}}$ be a basis of neighborhoods for x.

(i) For every $n \in \mathbb{N}$, let $U_n := \bigcap_{k=1}^{n} O_k$, and let x_n be any point in U_n. Show that $\{x_n\}_{n\in\mathbb{N}}$ converges to x.

(ii) Let $\{y_i\}_{i\in\mathbb{N}}$ be a sequence such that for every $n, k \in \mathbb{N}$ there is $i > n$ such that $y_i \in O_k$. Show that there exists a subsequence of $\{y_i\}_{i\in\mathbb{N}}$ that converges to x.

5.6. Non-first-countable topology on the real line ⚙. Let $\mathcal{T}$ be the family of subsets of the real line $\mathbb{R}$ defined in Problem 4.5, that is

$$\mathcal{T} := \emptyset \cup \{\mathbb{R} \setminus F \,:\, F \subset \mathbb{R} \text{ is finite}\}.$$

Show that $(\mathbb{R}, \mathcal{T})$ is not first-countable.

5.7. Separable and first-countable space ⚙. Let X be a separable first-countable topological space. Show that any dense subspace A of X is separable. Check that a metric space is first-countable, hence every dense subspace of a separable metric space is separable.

5.8. Separability and product ✎. Given two topological spaces X and Y, show that $X \times Y$ is separable if and only if both X and Y are separable.

5.9. L^∞ is not separable ⚙. Given a measurable function $f\colon [0,1] \to \mathbb{R}$ we define its *essential supremum* as

$$\operatorname{ess\,sup}|f| := \inf\{c \in [0,\infty] \,:\, |f| \le c \text{ almost everywhere}\}.$$

Consider the space

$$L^\infty([0,1]) := {\{f\colon [0,1] \to \mathbb{R} \,:\, f \text{ measurable}, \operatorname{ess\,sup}|f| < \infty\}}\big/{\sim},$$

where the relation $\sim$ is defined as $f \sim g$ if and only if $\operatorname{ess\,sup}|f - g| = 0$. Show that $L^\infty([0,1])$ equipped with the distance $d_\infty(f,g) := \operatorname{ess\,sup}|f - g|$, is not separable.

Note: On the other hand, for any $p \in [1,\infty)$, the space $L^p([0,1])$ equipped with the distance function $d_p(f,g) := \|f - g\|_p$ is separable. Indeed, a countable dense subset is given by

$$\left\{\sum_{i=k}^{n} a_k \chi_{I_k} \,:\, a_k \in \mathbb{Q},\ I_k \subseteq [0,1] \textit{ open interval with end points in } \mathbb{Q}\right\}.$$

5.10. Compact but not sequentially compact ⊛. Let $I = [0,1]$ and consider the space I^I. Show that I^I is compact but not sequentially compact.

Note: The space I^I is the product space of I-many copies of I, i.e. $\prod_{i\in I} I$, with the product topology.

5. Solutions

Solution of 5.1: Let $x = \{x_i\}_{i\in I}$ and $y = \{y_i\}_{i\in I}$ be two distinct points of X. In particular, there is a coordinate $j \in I$ such that $x_j \neq y_j$. Thus, we can find disjoint open sets U_j and V_j of X_j such that $x_j \in U_j$ and $y_j \in V_j$. Then define $U := \prod_{i\in I} \tilde{U}_i$ and $V := \prod_{i\in I} \tilde{V}_i$, where $\tilde{U}_i = \tilde{V}_i = X_i$ if $i \neq j$ and $\tilde{U}_j = U_j$, $\tilde{V}_j = V_j$. Observe that U and V are disjoint open subsets of X with $x \in U$ and $y \in V$, which proves that X is a Hausdorff space.

Solution of 5.2: Let $X = \{a, b, c, d\}$ be a set consisting of four points. Consider on X the following topology:

$$\mathcal{T} = \{\emptyset, \{a, b\}, \{c, d\}, X\}.$$

Note that every open set that contains a contains also b, thus X is not Hausdorff.

Other possible examples are the trivial topology $\mathcal{T} = \{\emptyset, X\}$ on any space X with more than one point and the cofinite topology on an infinite set (see Problem 4.5).

Solution of 5.3: Let x_1, x_2 be distinct points of X. We want to find disjoint open sets O_1 and O_2 such that $x_i \in O_i$, $i = 1, 2$. Since f is a bijection, $f(x_1)$ and $f(x_2)$ are distinct. Since Y is Hausdorff, there are disjoint open subsets U_1 and U_2 in Y such that $f(x_i) \in U_i$, $i = 1, 2$. Since f is continuous and bijective, $f^{-1}(U_1)$ and $f^{-1}(U_2)$ are disjoint open sets. Hence, setting $O_i = f^{-1}(U_i)$ provides the result we want.

Solution of 5.4: Let $y_1 \neq y_2$ be points of Y (and thus of X). Since X is Hausdorff, there are disjoint open sets O_1 and O_2 such that $y_i \in O_i$, $i = 1, 2$. Thus, $Y \cap O_1$ and $Y \cap O_2$ are disjoint open sets of Y that contain y_1 and y_2, respectively.

Solution of 5.5:

(i) Let V be an open set that contains x. Since $\{O_k\}_{k\in\mathbb{N}}$ is a basis of neighborhoods for x there exists $k \in \mathbb{N}$ such that $O_k \subseteq V$. In particular, for all $n \geq k$ we have that $U_n \subseteq V$. Hence $x_n \in V$ for every $n \geq k$, which proves that $\{x_n\}_{n\in\mathbb{N}}$ converges to x.

(ii) For every $m \in \mathbb{N}$, we define y_{i_m} as follows. Consider $U_m := \cap_{k=1}^m O_k$ defined in (i). Since U_m is an open set containing x, there exists $k_m \in \mathbb{N}$ such that $O_{k_m} \subseteq U_m$. Then we define y_{i_m} with $i_m > i_{m-1}$ such that $y_{i_m} \in O_{k_m}$, which exists by hypothesis. Then the result follows from part (i).

Solution of 5.6: Choose a point of $\mathbb{R}$, say 0, and suppose that there is a countable basis of neighborhoods $\{O_k\}_{k\in\mathbb{N}}$ for 0. We know that, for all $k \in \mathbb{N}$, O_k is of the form $\mathbb{R} \setminus F_k$ for some finite set $F_k \subseteq \mathbb{R}$. Then $F := \bigcup_{k=1}^\infty F_k$ is a countable union of finite sets, thus is countable. Since $\mathbb{R}$ is uncountable, there is a point $x \in \mathbb{R}$ different from 0 (indeed, uncountably many) that is not contained in F. Thus $U = \mathbb{R} \setminus \{x\}$ is an open set containing 0 that does not contain any O_k, for $k \in \mathbb{N}$, since $x \in O_k$ for all $k \in \mathbb{N}$. This contradicts the fact that $\{O_k\}_{k\in\mathbb{N}}$ is a countable basis of neighborhoods of 0.

Solution of 5.7: Consider a countable dense subset $\{x_n\}_{n\in\mathbb{N}}$ of X, which exists by separability of X. Then, since X is first-countable, for every $n \in \mathbb{N}$ there exists a basis of neighborhoods $\{B_{n,m}\}_{m\in\mathbb{N}}$ of x_n. Now, for every $n, m \in \mathbb{N}$, pick a point $y_{n,m} \in B_{n,m} \cap A$. This point exists since A is dense in X and $B_{n,m}$ is an open subset of X. We claim that $\{y_{n,m}\}_{n,m\in\mathbb{N}}$ is a countable dense subset of A. Consider any open set U of A. By definition of subspace topology, there exists an open set V of X such that $U = V \cap A$. Since $\{x_n\}_{n\in\mathbb{N}}$ is dense, there exists $n \in \mathbb{N}$ such that $x_n \in V$. Moreover, given that $\{B_{n,m}\}_{m\in\mathbb{N}}$ is a basis of neighborhoods of x_n, there is $m \in \mathbb{N}$ such that $B_{n,m} \subseteq V$. Now observe that $y_{n,m} \in B_{n,m} \cap A \subseteq V \cap A = U$, which proves that A is separable as we wanted.

Finally we prove that any metric space (X, d) is first-countable. Consider a point $x \in X$, then we claim that the family

$$\mathcal{B}_x := \{B(x, 1/k) \,:\, k \in \mathbb{N}_*\}$$

is a countable basis of neighborhoods of x (see also Problem 4.8 for a related proof). Consider any open set $U \subseteq X$ containing x. By definition of metric topology, there exists $y \in X$ and $r > 0$ such that $x \in B(y, r) \subseteq U$. Then observe that $B(x, r') \subseteq B(y, r) \subseteq U$, where $r' := r - d(x, y)$. Indeed, using the triangle inequality, for any $z \in B(x, r')$ it holds $d(y, z) \le d(y, x) + d(x, z) < d(x, y) + r' = r$. Now it is sufficient to pick some $k \in \mathbb{N}_*$ such that $1/k < r'$ to have that $B(x, 1/k) \subseteq U$, which prove that $\mathcal{B}_x$ is a basis of neighborhoods of x, since $B(x, 1/k) \in \mathcal{B}_x$.

Solution of 5.8: First, let us assume that $X \times Y$ is separable, i.e. there exists $\{(x_n, y_n)\}_{n\in\mathbb{N}} \subseteq X \times Y$ that is dense. We want to prove that $\{x_n\}_{n\in\mathbb{N}} \subseteq X$ and $\{y_n\}_{n\in\mathbb{N}} \subseteq Y$ are dense. Consider any point $x \in X$ and any open neighborhood $U \subseteq X$ of x. Then there exists $n \in \mathbb{N}$ such that $(x_n, y_n) \in U \times Y$, which implies that $x_n \in U$. This proves that $\{x_n\}_{n\in\mathbb{N}}$ is dense and thus X is separable. The result for Y is completely analogous.

Viceversa, let us assume that X and Y are separable, i.e. there exist contable dense subsets $\{x_n\}_{n\in\mathbb{N}} \subseteq X$ and $\{y_m\}_{m\in\mathbb{N}} \subseteq Y$. We want to prove that the countable set $\{(x_n, y_m)\}_{n,m\in\mathbb{N}}$ is dense in $X \times Y$. Consider any point $(x, y) \in X \times Y$ and any open neighborhood O of this point. Then there exists $U \times V \subseteq O$ that contains (x, y), where $U \subseteq X$ and $V \subseteq Y$ open sets; indeed $\{U \times V \subseteq X \times Y \,:\, U \subseteq X,\, V \subseteq Y \text{ open sets}\}$ is a basis for the product topology. Hence, since $\{x_n\}_{n\in\mathbb{N}} \subseteq X$ and $\{y_m\}_{m\in\mathbb{N}} \subseteq Y$ are dense, there are $n, m \in \mathbb{N}$ such that $x_n \in U$ and $y_m \in V$. This proves that $(x_n, y_m) \in U \times V \subseteq O$ and thus that $\{(x_n, y_m)\}_{n,m\in\mathbb{N}}$ is dense in $X \times Y$.

Solution of 5.9: For any $x \in [0, 1]$, let us denote by $f_x \in L^\infty([0, 1])$ the characteristic function of $[0, x) \subseteq [0, 1]$, i.e. $f_x := \chi_{[0,x)}$. Then, for every $0 \le x < y \le 1$, we have that

$$d_\infty(f_x, f_y) = \operatorname{ess\,sup} |f_x - f_y| = \operatorname{ess\,sup}\, \chi_{[x,y)} = 1.$$

Hence, the open sets of the family $\mathcal{B} := \{B(f_x, 1/2) \,:\, x \in [0,1]\}$ are pairwise disjoint. However, note that $\mathcal{B}$ has uncountably many elements and this proves that $L^\infty([0,1])$ cannot be separable.

Solution of 5.10: