6. Metric spaces

Chef's table

In this problem set, we want to test all general topological notions we have so far acquired in the specific context of metric spaces. As you can see, important metric spaces naturally arise as *functional spaces* (i.e. spaces whose points are functions). In the set below, the most significant exercises are perhaps the first two, 6.1 and 6.2, that lead to a complete proof (at the greatest possible level of generality) of the Arzelà-Ascoli compactness theorem, which is arguably one of the most useful compactness results in Mathematics and the basis for a number of advanced results.

At a conceptual level, we also investigate the question of determining when a distance (on a vector space) is determined from a norm, see Problem 6.8, and give a first glance at the way finite-dimensional vector spaces differ from infinite-dimensional ones, which is the real theme behind problems 6.5 and 6.6.

So, in this problem set you see three category of objects:

(i) topological spaces

(ii) metric spaces

(iii) normed linear spaces

each class being, in a natural way, included in the previous one (with two *proper* inclusions). The results you prove here will come back again and again along your mathematical path.

6.1. Space of continuous functions ⚙. Let (X, d_X) be a compact metric space and (Y, d_Y) be a complete metric space. Consider the space of continuous functions from X to Y, denoted by $C(X,Y) := \{f : X \to Y \ : \ f \text{ is continuous}\}$. We define a distance in $C(X,Y)$ as $d(f,g) := \sup_{x\in X} d_Y(f(x), g(x))$ for all $f, g \in C(X,Y)$.

(i) Show that d is indeed a metric on $C(X,Y)$.

(ii) Show that $C(X,Y)$ with distance d is a complete metric space.

6.2. Arzelà-Ascoli theorem ⚙. Consider a compact metric space (X, d_X) and a complete metric space (Y, d_Y). Then, as in Problem 6.1, consider the metric space $(C(X,Y), d)$ of continuous functions from X to Y. Prove that a subset $\mathcal{F} \subseteq C(X,Y)$ is *relatively compact* (i.e. $\overline{\mathcal{F}}$ is compact) if and only if it is

- *pointwise relatively compact*, i.e. $\mathcal{F}_x := \{f(x) \ : \ f \in \mathcal{F}\}$ is relatively compact in Y for all $x \in X$, and
- *equicontinuous*, i.e. for every $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ such that $d_Y(f(x), f(y)) < \varepsilon$ for all $f \in \mathcal{F}$ and for all $x, y \in X$ with $d_X(x,y) < \delta$.

6.3. Two sequences in a metric space ✍. Let $\{x_n\}_{n\in\mathbb{N}}$ and $\{y_n\}_{n\in\mathbb{N}}$ be two Cauchy sequences in a metric space (X,d). Prove that the sequence $\{d(x_n,y_n)\}_{n\in\mathbb{N}} \subseteq \mathbb{R}$ converges. Deduce that the map $d\colon X\times X\to\mathbb{R}$ is continuous with respect to the product topology.

6.4. Subset of complete is complete ✍. Let X be a complete metric space and let Y be a subset of X. Show that Y is complete if and only if it is closed.

6.5. Norms in a finite-dimensional space ⚙. Given a vector space X over $\mathbb{R}$, we say that two norms $\|\cdot\|$ and $\|\cdot\|'$ on X are *equivalent* if

$$\exists C>0: \quad C^{-1}\|x\|' \le \|x\| \le C\|x\|' \quad \forall x\in X.$$

Show that, if X is finite-dimensional, all norms on X are equivalent.

6.6. Non-equivalent distances and norms ⚙. Similarly to the case of norms (see Problem 6.5), we say that two metrics d and d' on a set X are *equivalent* if

$$\exists C>0: \quad C^{-1}d'(x_1,x_2) \le d(x_1,x_2) \le Cd'(x_1,x_2) \quad \forall x_1,x_2\in X.$$

(i) Construct two metrics on $\mathbb{R}^2$ that are *not* equivalent.

(ii) Construct a vector space X with two norms $\|\cdot\|$ and $\|\cdot\|'$ that are *not* equivalent.

Hint: Prove that $\|\cdot\|$ and $\|\cdot\|'$ are not equivalent by exhibiting a sequence $\{x_n\}_{n\in\mathbb{N}}\subseteq X$ that converges for $\|\cdot\|$ but not for $\|\cdot\|'$.

6.7. p-adic distance ✍. Let p be a prime number. Prove that the p-adic distance $d_p\colon \mathbb{Z}\times\mathbb{Z}\to[0,\infty)$ defined in Problem 2.9 is not equivalent to the Euclidean distance $d(x,y):=|x-y|$.

6.8. Metric induced by norm ✍. Let V be a vector space over $\mathbb{R}$. Show that a metric d on V is induced by a norm $\|\cdot\|$ (i.e. there exists a norm $\|\cdot\|$ such that $d(x,y)=\|x-y\|$ for all $x,y\in V$) if and only if the metric is

- *translation invariant*, i.e. $d(x+v,y+v)=d(x,y)$ for all $x,y\in V$ and $v\in V$, and
- *homogeneous*, i.e. $d(\lambda x,\lambda y)=|\lambda|d(x,y)$ for all $x,y\in V$ and $\lambda\in\mathbb{R}$.

6.9. A metric on $C^0(\mathbb{R}^m)$ ⚙. Let $K_1\subset K_2\subset\ldots\subset\mathbb{R}^m$ be a family of compact subsets such that $K_n\subset\operatorname{int}(K_{n+1})$ for every $n\in\mathbb{N}$ and $\bigcup_{n\in\mathbb{N}}K_n=\mathbb{R}^m$.

(i) Prove that

$$d(f,g) := \sum_{n\in\mathbb{N}} \frac{2^{-n}\|f-g\|_{C^0(K_n)}}{1+\|f-g\|_{C^0(K_n)}}$$

defines a metric d on $C^0(\mathbb{R}^m)$.

(ii) Show that $(C^0(\mathbb{R}^m), d)$ is complete.

(iii) Show that $C^0_c(\mathbb{R}^m)$, the space of continuous functions with compact support, is dense in $(C^0(\mathbb{R}^m), d)$.

Note: The same conclusions with the same proofs also hold for any *open set* $\Omega \subseteq \mathbb{R}^m$ *in place of* $\mathbb{R}^m$*. The metric* d *deals with the fact that* $C^0(\mathbb{R}^m)$ *contains unbounded functions like* $f(x) = |x|^2$ *for which* $\sup_{x\in\mathbb{R}^m}|f(x)| = \infty$*.*

6.10. Expanding map on a compact metric space. Let (X, d) be a compact metric space and let $f : X \to X$ be an expanding map, i.e. $d(f(x), f(y)) \geq d(x, y)$ for all $x, y \in X$. Prove that f is an isometry, i.e. $d(f(x), f(y)) = d(x, y)$ for all $x, y \in X$.

6. Solutions

Solution of 6.1:

(i) Let us check the properties on d to be a metric.

- $0 \leq d(f,g) < \infty$: Obviously $d(f,g) \geq 0$ for all $f, g \in C(X,Y)$, since $d_Y(f(x), g(x)) \geq 0$ for all $x \in X$. Moreover observe that $x \mapsto d_Y(f(x), g(x))$ is a continuous function from X to $\mathbb{R}$, hence its supremum is attained at some point $x_0 \in X$, by compactness of X. Therefore we have $d(f,g) = d_Y(f(x_0), g(x_0)) < \infty$.
- $d(f,g) = 0 \iff f = g$: It is clear that if $f = g$, then $d(f,g) = 0$. Viceversa, assume that $d(f,g) = 0$, then $\sup_{x\in X} d_Y(f(x), g(x)) = 0$. Thus, for every $x \in X$ we have $d_Y(f(x), g(x)) = 0$. Since d_Y is a metric on Y, we obtain $f(x) = g(x)$, for all $x \in X$.
- $d(f,g) = d(g,f)$: This is clear since d_Y is symmetric.
- $d(f,g) + d(g,h) \geq d(f,h)$: Using that the triangle inequality holds for d_Y on Y, we directly obtain

$$\begin{aligned} d(f,g) + d(g,h) &= \sup_{x\in X} d_Y(f(x), g(x)) + \sup_{x\in X} d_Y(g(x), h(x)) \\ &\geq \sup_{x\in X} \{d_Y(f(x), g(x)) + d_Y(g(x), h(x))\} \\ &\geq \sup_{x\in X} d_Y(f(x), h(x)) = d(f,h). \end{aligned}$$

(ii) Let $\{f_n\}_{n\in\mathbb{N}}$ be a Cauchy sequence in $C(X,Y)$. Then, for all $x \in X$, the sequence $\{f_n(x)\}_{n\in\mathbb{N}}$ is a Cauchy sequence in Y, because $d_Y(f_n(x), f_m(x)) \leq d(f_n, f_m)$ for all $n, m \in \mathbb{N}$. Thus, the sequence $\{f_n(x)\}_{n\in\mathbb{N}}$ converges to a point $f(x) \in Y$. We claim that the function f defined in this way is the limit of the sequence $\{f_n\}_{n\in\mathbb{N}}$ in $C(X,Y)$.

First observe that, for every $x, y \in X$ and $n \in \mathbb{N}$, by triangle inequality and definition of f it holds that

$$\begin{aligned} d_Y(f(x), f(y)) &\leq d_Y(f(x), f_n(x)) + d_Y(f_n(x), f_n(y)) + d_Y(f_n(y), f(y)) \\ &= \lim_{m\to\infty} d_Y(f_m(x), f_n(x)) + d_Y(f_n(x), f_n(y)) + \lim_{m\to\infty} d_Y(f_n(y), f_m(y)) \\ &\leq d_Y(f_n(x), f_n(y)) + 2 \lim_{m\to\infty} d(f_n, f_m). \end{aligned}$$

From this inequality follows that f is continuous. Indeed, consider a sequence $\{x_k\}_{k\in\mathbb{N}} \subseteq X$ converging to $x_0 \in X$. For every $\varepsilon > 0$ there is $M > 0$ such that $d(f_n, f_m) < \varepsilon/2$ for all $n, m \geq M$. Then, for all $n \geq M$ and $x, y \in X$, $d_Y(f(x), f(y)) \leq d_Y(f_n(x), f_n(y)) + \varepsilon$. In particular, since f_n is continuous, choosing $x = x_0$ and $y = x_k$, we get

$$\liminf_{k\to\infty} d_Y(f(x_0), f(x_k)) \leq \lim_{k\to\infty} d_Y(f_n(x_0), f_n(x_k)) + \varepsilon = \varepsilon.$$

Since this holds for every $\varepsilon > 0$, we obtain $\liminf_{k\to\infty} d_Y(f(x_0), f(x_k)) = 0$, which proves that f is continuous.

Let us now prove that $d(f, f_n) \to 0$ as $n \to \infty$. As before, for every $\varepsilon > 0$ there is $M > 0$ such that $d(f_n, f_m) < \varepsilon$ for all $n, m > M$. Hence, for $n > M$ and x_0 realizing $d(f, f_n) = d_Y(f(x_0), f_n(x_0))$ (which exists as observed in the proof of part (i)), we have that

$$d(f, f_n) = d_Y(f(x_0), f_n(x_0)) = \lim_{m\to\infty} d_Y(f_m(x_0), f_n(x_0)) \le \lim_{m\to\infty} d(f_m, f_n) \le \varepsilon,$$

which implies that $\lim_{n\to\infty} d(f, f_n) = 0$, as we wanted.

Solution of 6.2: First, let us assume that $\mathcal{F}$ is relatively compact and prove that it is pointwise relatively compact and equicontinuous.

- $\mathcal{F}$ *pointwise relatively compact.* Consider a sequence $\{f_n(x)\}_{n\in\mathbb{N}} \subseteq \mathcal{F}_x$. Since $\mathcal{F}$ is relatively compact, there exists a subsequence $\{f_{n_k}\}_{k\in\mathbb{N}}$ of $\{f_n\}_{n\in\mathbb{N}} \subseteq \mathcal{F}$ that converges to some $f \in \overline{\mathcal{F}}$. Then observe that $d_Y(f(x), f_{n_k}(x)) \le d(f, f_{n_k}) \to 0$ as $k \to \infty$. Hence, $\{f_n(x)\}_{n\in\mathbb{N}}$ admits a converging subsequence in $(\overline{\mathcal{F}})_x \subseteq \overline{\mathcal{F}_x}$ (i.e. $\{f_{n_k}(x)\}_{k\in\mathbb{N}}$) and thus we proved that $\mathcal{F}$ is pointwise relatively compact.
- $\mathcal{F}$ *equicontinuous.* Assume by contradiction that $\mathcal{F}$ is not equicontinuous, then there is $\varepsilon > 0$ such that for all $n \in \mathbb{N}$ there exists $x_n, y_n \in X$ and $f_n \in \mathcal{F}$ with $d_X(x_n, y_n) < 1/n$ and $d_Y(f_n(x_n), f_n(y_n)) \ge \varepsilon$. Since X and $\overline{\mathcal{F}}$ are compact, up to subsequence we can assume that $x_n \to x$, $y_n \to y$ and $f_n \to f$ as $n \to \infty$, for some $x, y \in X$ and $f \in \overline{\mathcal{F}}$. Now observe that $d_X(x, y) = \lim_{n\to\infty} d_X(x_n, y_n) = 0$ (by Problem 6.3), hence $x = y$. On the other hand, $f_n(x_n) \to f(x)$ as $n \to \infty$; indeed

$$\begin{aligned} d_Y(f_n(x_n), f(x)) &\le d_Y(f_n(x_n), f(x_n)) + d_Y(f(x_n), f(x)) \\ &\le d(f_n, f) + d_Y(f(x_n), f(x)) \to 0, \end{aligned}$$

  since $f_n \to f$, $x_n \to x$ and f is continuous. Therefore we obtain (again by Problem 6.3) that $d_Y(f(x), f(y)) = \lim_{n\to\infty} d_Y(f_n(x_n), f_n(y_n)) \ge \varepsilon$, hence $f(x) \ne f(y)$, which contradicts $x = y$.

Viceversa, let us assume that $\mathcal{F}$ is pointwise relatively compact and equicontinuous. We want to prove that $\overline{\mathcal{F}}$ is compact. First recall that, being a compact metric space, X is separable (see e.g. proof of (C2) $\Longrightarrow$ (C1) in Problem 4.9), i.e. there exists a sequence $D = \{x_k\}_{k\in\mathbb{N}}$ that is dense in X. Now consider any sequence $\{f_n\}_{n\in\mathbb{N}}$, we want to prove that it admits a converging subsequence. We start by constructing a subsequence that pointwise converges on D.

Since $\mathcal{F}$ is pointwise relatively compact, there exists a subsequence $\{f_n^{(0)}\}_{n\in\mathbb{N}}$ of $\{f_n\}_{n\in\mathbb{N}}$ that pointwise converges on x_0, i.e. $f_n^{(0)}(x_0) \to y_0$ for some $y_0 \in Y$. Similarly, for every $k \in \mathbb{N}$, there is a subsequence $\{f_n^{(k+1)}\}_{n\in\mathbb{N}}$ of $\{f_n^{(k)}\}_{n\in\mathbb{N}}$ such that $f_n^{(k+1)}(x_{k+1}) \to y_{k+1}$ for some $y_{k+1} \in Y$. Hence, let us consider (by a standard diagonal argument) the sequence $\{f_m^{(m)}\}_{m\in\mathbb{N}}$, which in particular is a subsequence of the initial sequence $\{f_n\}_{n\in\mathbb{N}}$. Observe that $\{f_m^{(m)}\}_{m\in\mathbb{N}}$ pointwise converges on D. We want to prove that $\{f_m^{(m)}\}_{m\in\mathbb{N}}$ is in fact a Cauchy sequence in $(C(X,Y), d)$ and thus converges.

For any $\varepsilon > 0$, consider $\delta = \delta(\varepsilon) > 0$ given by the equicontinuity assumption on $\mathcal{F}$. By compactness of X, the open cover $\{B_\delta(x_k)\}_{k\in\mathbb{N}}$ admits a finite subcover $\{B_\delta(x_{k_1}), \dots, B_\delta(x_{k_l})\}$. Hence, for any $x \in X$, there is $i = 1, \dots, l$ such that $x \in B_\delta(x_{k_i})$. From this it follows that

$$\begin{aligned} d_Y(f_m^{(m)}(x), f_{m'}^{(m')}(x)) &\leq d_Y(f_m^{(m)}(x), f_m^{(m)}(x_i)) + d_Y(f_m^{(m)}(x_i), f_{m'}^{(m')}(x_i)) + \\ &+ d_Y(f_{m'}^{(m')}(x_i), f_{m'}^{(m')}(x)) \leq 2\varepsilon + d_Y(f_m^{(m)}(x_i), f_{m'}^{(m')}(x_i)). \end{aligned}$$

However, recall that $\{f_m^{(m)}\}_{m\in\mathbb{N}}$ pointwise converges on $x_{k_1}, \dots, x_{k_l}$. Hence, there is $M \geq 0$ such that $d_Y(f_m^{(m)}(x_i), f_{m'}^{(m')}(x_i)) \leq \varepsilon$ for all $i = 1, \dots, l$ and for all $m, m' \geq M$. This implies that $d_Y(f_m^{(m)}(x), f_{m'}^{(m')}(x)) \leq 3\varepsilon$ for all $x \in X$ and for all $m, m' \geq M$ and thus proves that $\{f_m^{(m)}\}_{m\in\mathbb{N}}$ is a Cauchy sequence as we wanted.

Solution of 6.3: We will prove that $\{d(x_n, y_n)\}_{n\in\mathbb{N}}$ is a Cauchy sequence, from which follows that it converges, because $\mathbb{R}$ is complete. For every $\varepsilon > 0$, since $\{x_n\}_{n\in\mathbb{N}}$ and $\{y_n\}_{n\in\mathbb{N}}$ are Cauchy sequences, there exists $N > 0$ such that $d(x_n, x_m) < \varepsilon/2$ and $d(y_n, y_m) < \varepsilon/2$ for all $n, m \geq N$. Hence, for $n, m \geq N$, we have that

$$\begin{aligned} |d(x_n, y_n) - d(x_m, y_m)| &= |d(x_n, y_n) - d(x_m, y_n) + d(x_m, y_n) - d(x_m, y_m)| \\ &\leq |d(x_n, y_n) - d(x_m, y_n)| + |d(x_m, y_n) - d(x_m, y_m)| \\ &\leq d(x_n, x_m) + d(y_n, y_m) < \varepsilon. \end{aligned}$$

This proves that the sequence $\{d(x_n, y_n)\}_{n\in\mathbb{N}}$ is Cauchy, as we wanted.

Solution of 6.4: Suppose that Y is not complete. Then there is a Cauchy sequence $\{x_n\}_{n\in\mathbb{N}} \subseteq Y$ that does not admit a limit in Y. Since X is complete, $\{x_n\}_{n\in\mathbb{N}}$ converges to some $x \in X$. Thus $x \in X \setminus Y$ and $x \in \overline{Y}$, which means that Y is not closed in X.

On the other hand, suppose that Y is complete. Thus every Cauchy sequence admits a limit in Y. However, each converging sequence in a metric space is a Cauchy sequence, thus $Y = \overline{Y}$.

Solution of 6.5: Let $n = \dim X$ and let $\{e_1, \dots, e_n\}$ be a basis for X. Then every $x \in X$ is of the form $x = \sum_{k=1}^n x_k e_k$ with uniquely determined components $x_1, \dots, x_n \in \mathbb{R}$. Recall that

$$\|x\|_\infty := \max_{k\in\{1,\dots,n\}} |x_k|$$

defines a norm on X. We show that any given norm $\|\cdot\|$ is equivalent to $\|\cdot\|_\infty$ and therefore any two norms are equivalent to each other. We have

$$\begin{aligned} \|x\| = \Big\|\sum_{k=1}^n x_k e_k\Big\| &\leq \sum_{k=1}^n \|x_k e_k\| = \sum_{k=1}^n |x_k| \|e_k\| \\ &\leq n\Big(\max_{k\in\{1,\dots,n\}} |x_k|\Big)\Big(\max_{k\in\{1,\dots,n\}} \|e_k\|\Big) = nM\|x\|_\infty \qquad (*) \end{aligned}$$

where

$$M := \left(\max_{k\in\{1,\dots,n\}} \|e_k\|\right)$$

is a finite constant. The triangle inequality implies $\Big|\|x\| - \|y\|\Big| \leq \|x-y\|$. Combined with $(*)$ we have

$$\Big|\|x\| - \|y\|\Big| \leq \|x-y\| \leq nM\|x-y\|_\infty$$

for every $x, y \in X$. This implies that $\|\cdot\|\colon (X, \|\cdot\|_\infty) \to \mathbb{R}$ is a continuous map. We restrict this map to $K := \{x \in X : \|x\|_\infty = 1\}$. Note that K is a closed and bounded subset of $(X, \|\cdot\|_\infty)$. Moreover recall that $\|\cdot\|_\infty$ is equivalent to the Euclidean norm $\|\cdot\|_2$, which is defined as $\|x\|_2 := (\sum_{k=1}^n |x_k|^2)^{1/2}$ for every $x = \sum_{k=1}^n x_k e_k \in X$ (in particular $\|x\|_\infty \leq \|x\|_2 \leq n\|x\|_\infty$). Hence, by Heine-Borel theorem, K is compact. Therefore, the function $\|\cdot\|$ attains minimum and maximum on K, i.e. there exists $x_1, x_2 \in X$ such that

$$m_1 := \min_{x\in K}\|x\| = \|x_1\|, \qquad\qquad m_2 := \max_{x\in K}\|x\| = \|x_2\|.$$

Since $\|x_1\|_\infty = 1$ we have $x_1 \neq 0$ and $m_1 > 0$. Then, for an arbitrary $x \in X \setminus \{0\}$ we have

$$\frac{x}{\|x\|_\infty} \in K \quad \implies \quad 0 < m_1 \leq \left\|\frac{x}{\|x\|_\infty}\right\| \leq m_2 < \infty.$$

Multiplication with $\|x\|_\infty$ implies

$$0 < m_1\|x\|_\infty \leq \|x\| \leq m_2\|x\|_\infty < \infty.$$

Any other given norm $\|\cdot\|'$ satisfies analogously

$$0 < m_1'\|x\|_\infty \leq \|x\|' \leq m_2'\|x\|_\infty < \infty.$$

Thus, the combination of the two last inequalities proves the equivalence of $\|\cdot\|$ and $\|\cdot\|'$.

Solution of 6.6:

(i) Let d be the metric on $\mathbb{R}^2$ induced by the Euclidean norm. We define d' on $\mathbb{R}^2$ by

$$d'(x,y) = \begin{cases} 0 & \text{if } x = y, \\ 1 & \text{if } x \neq y. \end{cases}$$

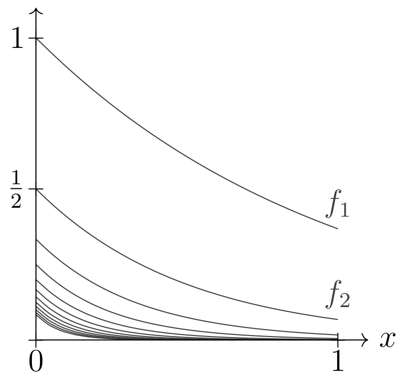
Let z be a point on the Euclidean unit circle $\mathbb{S}^1 \subset \mathbb{R}^2$ let $z_n = \frac{1}{n}z$. Then, $d(0, z_n) = \frac{1}{n}$ and $d'(0, z_n) = 1$. Since an inequality of the form $1 \leq C/n$ cannot hold for every $n \in \mathbb{N}$ if C is finite, d and d' are not equivalent.

(ii) Let $X = C^1([0,1])$. Let $\|\cdot\|$ and $\|\cdot\|'$ be the two norms on X given by

$$\|u\| := \|u\|_{C^0} = \sup_{x\in[0,1]} |u(x)|, \qquad \|u\|' := \max\left\{\sup_{x\in[0,1]} |u(x)|, \sup_{x\in[0,1]} |u'(x)|\right\}$$

For $n \in \mathbb{N}$ we consider

$$\begin{aligned} f_n \colon [0,1] &\to \mathbb{R} \\ x &\mapsto \frac{e^{-nx}}{n}. \end{aligned}$$



Then, $f_n \in C^1([0,1])$ for every $n \in \mathbb{N}$. Moreover, $\|f_n\| = \frac{1}{n}$ and $\|f_n\|' = 1$. Since an inequality of the form $1 \leq C/n$ cannot hold for every $n \in \mathbb{N}$ if C is finite, $\|\cdot\|$ and $\|\cdot\|'$ are not equivalent.

Solution of 6.7: For all $m \in \mathbb{N}$, consider the numbers $x = p^m$ and $y = 2p^m$. Then $d(x,y) = p^m$ and $d_p(x,y) = p^{-m}$. In particular we have that

$$\frac{d(x,y)}{d_p(x,y)} = p^{2m} \to \infty \quad \text{as } m \to \infty,$$

thus d and d_p are not equivalent.

Solution of 6.8: If the metric d is induced by the norm $\|\cdot\|$, then

$$d(x+v, y+v) = \|(x+v) - (y+v)\| = \|x-y\| = d(x,y),$$
$$d(\lambda x, \lambda y) = \|\lambda x - \lambda y\| = \|\lambda(x-y)\| = |\lambda|\|x-y\|.$$

Conversely, we assume that the metric d is translation invariant and homogeneous and claim that $\|x\| := d(x,0)$ defines a norm which induces d. The function $\|\cdot\|$ is indeed a norm, because for all $x, y, z \in V$ and $\lambda \in \mathbb{R}$, we have

- $\|x\| = 0 \iff d(x,0) = 0 \iff x = 0,$
- $\|\lambda x\| = d(\lambda x, 0) = d(\lambda x, \lambda 0) = |\lambda| d(x,0) = |\lambda|\|x\|,$
- $\|x+y\| = d(x+y, 0) \leq d(x+y, y) + d(y,0) = d(x,0) + d(y,0) = \|x\| + \|y\|.$

Moreover, $\|\cdot\|$ induces the metric d since for all $x, y \in V$

$$\|x-y\| = d(x-y, 0) = d(x,y).$$

Solution of 6.9:

(i) The function $d\colon C^0(\mathbb{R}^m)\times C^0(\mathbb{R}^m)\to\mathbb{R}$ is well-defined because for any $f,g\in C^0(\mathbb{R}^m)$,

$$0\leq d(f,g)=\sum_{n\in\mathbb{N}}\frac{2^{-n}\|f-g\|_{C^0(K_n)}}{1+\|f-g\|_{C^0(K_n)}}\leq\sum_{n\in\mathbb{N}}2^{-n}<\infty.$$

The symmetry and the requirement $d(f,g)=0$ if and only if $f=g$, both follow from the respective property of $\|f-g\|_{C^0(K_n)}$. It remains to prove the triangle inequality. For every $a,b,c\geq 0$ with $a+b\leq c$, it holds that

$$\begin{aligned}\frac{c}{1+c}&=1-\frac{1}{1+c}\leq 1-\frac{1}{1+a+b}=\frac{a+b}{1+a+b}=\frac{a}{1+a+b}+\frac{b}{1+a+b}\\&\leq\frac{a}{1+a}+\frac{b}{1+b}.\end{aligned}$$

Applying this inequality with $a=\|f-g\|_{C^0(K_n)}, b=\|g-h\|_{C^0(K_n)}, c=\|f-h\|_{C^0(K_n)}$ for all $n\in\mathbb{N}$, we obtain

$$\begin{aligned}d(f,h)&=\sum_{n\in\mathbb{N}}\frac{2^{-n}\|f-h\|_{C^0(K_n)}}{1+\|f-h\|_{C^0(K_n)}}\leq\sum_{n\in\mathbb{N}}2^{-n}\left(\frac{\|f-g\|_{C^0(K_n)}}{1+\|f-g\|_{C^0(K_n)}}+\frac{\|g-h\|_{C^0(K_n)}}{1+\|g-h\|_{C^0(K_n)}}\right)\\&=d(f,g)+d(g,h),\end{aligned}$$

as we wanted.

(ii) In the following, the restriction of $f\in C^0(\mathbb{R}^m)$ to $K\subset\mathbb{R}^m$ is denoted by $f|_K$.

Claim 1. If $(f_k)_{k\in\mathbb{N}}$ is a Cauchy-sequence in $(C^0(\mathbb{R}^m),d)$, then $(f_k|_{K_n})_{k\in\mathbb{N}}$ is a Cauchy-sequence in $\big(C^0(K_n),\|\cdot\|_{C^0(K_n)}\big)$ for each of the compact sets $K_n\subset\mathbb{R}^m$.

Proof. Let $n_0\in\mathbb{N}$ be arbitrary but fixed and let $0<\varepsilon<1$. By assumption there exists $K\in\mathbb{N}$ such that for every $k,\ell>K$

$$d(f_k,f_\ell)=\sum_{n\in\mathbb{N}}\frac{2^{-n}\|f_k-f_\ell\|_{C^0(K_n)}}{1+\|f_k-f_\ell\|_{C^0(K_n)}}<\frac{\varepsilon}{2^{n_0}(1+\varepsilon)}.$$

In particular, since every summand is non-negative,

$$\frac{2^{-n_0}\|f_k-f_\ell\|_{C^0(K_{n_0})}}{1+\|f_k-f_\ell\|_{C^0(K_{n_0})}}<\frac{\varepsilon}{2^{n_0}(1+\varepsilon)}.$$

This implies $\|f_k|_{K_{n_0}}-f_\ell|_{K_{n_0}}\|_{C^0(K_{n_0})}=\|f_k-f_\ell\|_{C^0(K_{n_0})}<\varepsilon$ for every $k,\ell>K$ which means that $(f_k|_{K_{n_0}})_{k\in\mathbb{N}}$ is a Cauchy sequence. □

Claim 2. Let $(f_k)_{k\in\mathbb{N}}$ be a sequence in $C^0(\mathbb{R}^m)$ and let $f\in C^0(\mathbb{R}^m)$. If $f_k|_{K_n}\xrightarrow{k\to\infty}f|_{K_n}$ in $C^0(K_n)$ for every $n\in\mathbb{N}$, then $f_k\xrightarrow{k\to\infty}f$ in $(C^0(\mathbb{R}^m),d)$.

Proof. Let $\varepsilon>0$ and let $N_\varepsilon\in\mathbb{N}$ such that

$$\sum_{n=N_\varepsilon+1}^{\infty}2^{-n}=2^{-N_\varepsilon}\leq\frac{\varepsilon}{2}.$$

By assumption, there exists $K \in \mathbb{N}$ such that $\|f_k - f\|_{C^0(K_n)} = \|f_k|_{K_n} - f|_{K_n}\|_{C^0(K_n)} < \frac{\varepsilon}{2}$ for every $k \geq K$ and all the finitely many $n \in \{1, \ldots, N_\varepsilon\}$. Hence

$$\begin{aligned} d(f_k, f) &= \sum_{n=1}^{N_\varepsilon} \frac{2^{-n}\|f_k - f\|_{C^0(K_n)}}{1 + \|f_k - f\|_{C^0(K_n)}} + \sum_{n=N_\varepsilon+1}^{\infty} \frac{2^{-n}\|f_k - f\|_{C^0(K_n)}}{1 + \|f_k - f\|_{C^0(K_n)}} \\ &\leq \frac{\varepsilon}{2} \sum_{n=1}^{N_\varepsilon} 2^{-n} + \sum_{n=N_\varepsilon+1}^{\infty} 2^{-n} \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$ □

Let $(f_k)_{k\in\mathbb{N}}$ be a Cauchy-sequence in $(C^0(\mathbb{R}^m), d)$. Since $\big(C^0(K_n), \|\cdot\|_{C^0(K_n)}\big)$ is complete, $(f_k|_{K_n})_{k\in\mathbb{N}}$ has a limit $g_n \in C^0(K_n)$ for every $n \in \mathbb{N}$ by Claim 1. In particular, given any $n \in \mathbb{N}$ pointwise convergence $f_k(x) \to g_n(x)$ holds for every $x \in K_n$. Since the pointwise limit is unique, $g_{n+1}|_{K_n} = g_n$ for every $n \in \mathbb{N}$. Therefore, there exists a well-defined function $f\colon \mathbb{R}^m \to \mathbb{R}$ such that $g_n = f|_{K_n}$.

Because $K_n \subset \operatorname{int}(K_{n+1})$ and $\bigcup_{n\in\mathbb{N}} K_n = \mathbb{R}^m$, every point $x \in \mathbb{R}^m$ has a neighbourhood on which f inherits the continuity of g_n for some $n \in \mathbb{N}$. Thus, $f \in C^0(\mathbb{R}^m)$. Then, by Claim 2, $(f_k)_{k\in\mathbb{N}}$ converges to f in $(C^0(\mathbb{R}^m), d)$. Therefore, $(C^0(\mathbb{R}^m), d)$ is complete.

(iii) Let $f \in C^0(\mathbb{R}^m)$ be arbitrary. Since K_n is compact with $K_n \subset \operatorname{int}(K_{n+1})$, we have

$$\varepsilon_n := \operatorname{dist}\Big(K_n, (\operatorname{int}(K_{n+1}))^{\mathsf{c}}\Big) > 0.$$

For every $n \in \mathbb{N}$ we define the function $\varphi \in C^0_c(\mathbb{R}^m)$ by

$$\varphi_n(x) = \begin{cases} 1 - \frac{1}{\varepsilon_n}\operatorname{dist}(x, K_n) & \text{if } \operatorname{dist}(x, K_n) \leq \varepsilon_n \\ 0, & \text{else,} \end{cases}$$

and consider $f_k := \varphi_k f$. By construction, $f_k|_{K_n} \xrightarrow{k\to\infty} f|_{K_n}$ for every $n \in \mathbb{N}$. Therefore, $f_k \xrightarrow{k\to\infty} f$ in $(C^0(\mathbb{R}^m), d)$ by Claim 2 of (ii). Since $f_k \in C^0_c(\mathbb{R}^m)$ for every $k \in \mathbb{N}$ and since $f \in C^0(\mathbb{R}^m)$ was arbitrary, we have shown that $C^0_c(\mathbb{R}^m)$ is dense in $(C^0(\mathbb{R}^m), d)$.

Solution of 6.10: