7. Quotients

Chef's table

This problem set focuses on quotients, but actually provides an excellent resource for you to review (almost) all the material we have so far seen in the course before we move on to an introduction to Algebraic Topology.

In many respects, you may consider this assignment as an *unofficial* take-home midterm and we encourage you to do so. For this reason we have decided to give you two weeks to work on these exercises, and we have also refrained from adding a challenge problem.

Some more specific comments: the first five exercises are rather straightforward, have short solutions and are intended as a warm-up. Among them, Problem 7.3 is close in spirit to Part Ib of the exam paper. Problems 7.6-7.7 are meant to help you acquire a working knowledge on quotients in some (fundamental!) examples, while 7.8 is a standard fact on the Alexandroff compactification (which will be presented during the next exercise class). Lastly, problems 7.9-7.10 collect some additional facts about quotients (which are important for you to keep in mind). Either of them is also close to Part IIa of the exam paper. Problem 7.10 is by far the most demanding exercise in this set, although it is supposed to be tractable by all students.

7.1. Important counterexample ✍. Consider the equivalence relation $\sim$ on $\mathbb{R}$, given by $x \sim y$ if and only if

$$x = y \qquad \text{or} \qquad |x| = |y| \text{ and } |x| > 1.$$

Let $Y := \mathbb{R}/\sim$ equipped with the induce topology. Show that Y is not a Hausdorff space.

7.2. Quotients and non-quotients ✍. Show that there is a quotient map $q\colon (-2, 2) \to [-1, 1]$, but not a quotient map $p\colon [-2, 2] \to (-1, 1)$.

7.3. Properties that descends to the quotient (or not) ✍. Let X, Y be topological spaces, and let $f\colon X \to Y$ be a continuous surjection. Assume that Y is equipped with the quotient topology, which means that a set $U \subseteq Y$ is open if and only if $f^{-1}(U)$ is open in X. Decide if the following statements are true or false: in case they are true, prove them; in case they are false, find a counterexample.

(i) If X is compact, so it is Y.

(ii) If X is Hausdorff, so it is Y.

(iii) If X is normal, then Y is Hausdorff.

(iv) If $|X| = \infty$, then $|Y| = \infty$.

(v) If X is connected, so it is Y.

(vi) If X is a metric space, so it is Y.

7.4. Connected components and quotients. Let X be a topological space, and assume that all connected components of X are open. Let $q\colon X \to Y$ be a quotient of X. Show that the connected components of Y are also open.

Note: It is not always true that all connected components are open, an example is $\mathbb{Q} \subseteq \mathbb{R}$ with the induced topology.

7.5. Saturated subsets. Let $q\colon X \to Y$ be a quotient. We say that a subset $A \subseteq X$ is *saturated* if $q^{-1}(q(A)) = A$. Show that if q is an open map and $A \subseteq X$ is a saturated subset, then also $\overline{A}$ and $\mathrm{int}(A)$ are saturated. Give an example where the map q is not open and the above is false (i.e. there is a saturated set B such that $\overline{B}$ or $\mathrm{int}(B)$ are not saturated).

7.6. The torus. Prove that the topological product space $X_1 = S^1 \times S^1$ is homeomorphic to the quotient $X_2 = Q/\sim$ obtained by considering on $Q = [0,1]^2$ the equivalence relation $(s,0) \sim (s,1)$ for all $s \in [0,1]$, $(0,t) \sim (1,t)$ for all $t \in [0,1]$. These are two possible (equivalent) definition of the torus T^2.

7.7. The real projective space. Prove that the topological spaces X_1, X_2 and X_3 defined below are homeomorphic. These are indeed three possible (equivalent) definitions of the projective space $\mathbb{P}^2(\mathbb{R})$ (see Lecture 12).

(i) Let $S^2 \subseteq \mathbb{R}^3$ be the two-dimensional unit sphere and consider on S^2 the relation $\sim$ that identifies the antipodal points on S^2, i.e. $u \sim v$ if and only if $u = v$ or $u = -v$. Then X_1 is defined as the quotient $X_1 := S^2/\sim$.

(ii) Consider the two-dimensional unit disk $D^2 \subseteq \mathbb{R}^2$ and the relation $\simeq$ on D^2 that identifies the antipodal points on its boundary, i.e. $u \simeq v$ if and only if $u = v$ or $u = -v$ with $u, v \in \partial D^2$. The topological space X_2 is then the quotient $X_2 := D^2/\simeq$.

(iii) Let $\mathcal{L}$ be the set of lines in $\mathbb{R}^3$ passing through the origin. Given $L_1, L_2 \in \mathcal{L}$, let $0 \leq \alpha \leq \pi/2$ be the angle between L_1 and L_2 and define $d(L_1, L_2) := \alpha$. Then we define the topological space X_3 as $\mathcal{L}$ with the topology induced by d.

7.8. Compactification of the Euclidean space. Prove that the Alexandroff one-point compactification of $\mathbb{R}^n$ is homeomorphic to S^n for all $n \geq 1$.

7.9. Hausdorff quotient. Let X be a topological space and let $\Delta := \{(x,y) \in X \times X \,:\, x = y\}$ be the diagonal of $X \times X$.

(i) Show that X is Hausdorff if and only in Δ is closed in $X \times X$.

(ii) Let $\sim$ be any equivalence relation on X and define $R := \{(x, y) \in X \times X \,:\, x \sim y\}$. Suppose that $q\colon X \to X/{\sim}$ is open. Show that $X/{\sim}$ is Hausdorff if and only if R is closed in $X \times X$.

7.10. Quotient with respect to a compact set ⚙. Let X be a Hausdorff space and let K be a compact subset of X. Show that:

(i) The quotient X/K is Hausdorff.

Note: The notation X/K is standard to indicate $X/{\sim}$, where the equivalence relation $\sim$ is given by $x \sim y$ if and only if $x = y$ or $x, y \in K$.

(ii) Let A be an open subset of X strictly contained in K (i.e. $A \subsetneq K$). Then the map $f\colon (X \setminus A)/(K \setminus A) \to X/K$ defined as $f([x]_A) := [x]$ is well-defined and a homeomorphism. Here we denote with $[x]_A$ and $[x]$ the equivalence classes in $(X \setminus A)/(K \setminus A)$ and X/K, respectively.

(iii) Item (ii) is not true if $A = K$.

(iv) Assume that X is compact. Then X/K is the Alexandroff one-point compactification of $X \setminus K$.

7. Solutions

Solution of 7.1: Let $q\colon \mathbb{R} \to Y$ be the quotient map. Note that $q(1) \neq q(-1)$. Let U and V be open sets of Y such that $q(1) \in U$ and $q(-1) \in V$. We claim that U and V must intersect. The counterimages $q^{-1}(U)$ and $q^{-1}(V)$ are open sets of $\mathbb{R}$ that contain 1 and -1 respectively. In particular, there are $\varepsilon_1, \varepsilon_2 > 0$ such that $(1-\varepsilon_1, 1+\varepsilon_1) \subseteq q^{-1}(U)$ and $(-1-\varepsilon_2, -1+\varepsilon_2) \subseteq q^{-1}(V)$. In particular, taking $\varepsilon < \min\{\varepsilon_1, \varepsilon_2\}$, we have that $1+\varepsilon \in q^{-1}(U)$ and $-1-\varepsilon \in q^{-1}(V)$. However, since $q(1+\varepsilon) = q(-1-\varepsilon)$, we have that U and V intersects, which concludes the proof.

Solution of 7.2: Let $\sim$ the equivalence relation on $(-2,2)$ given by

$$x \sim y \iff x = y \text{ or } x, y \leq -1 \text{ or } x, y \geq 1.$$

Then $(-2,2)/{\sim}$ is homeomorphic to $[-1,1]$. On the other hand, if there were a continuous surjection $p\colon [-2,2] \to (-1,1)$, we would have that the latter is compact, which is a contradiction.

Solution of 7.3:

(i) True, because Y is the continuous image of a compact set.

(ii) False. A counterexample is given by Problem 7.1. Indeed, the space Y in the problem is not Hausdorff, even though it is quotient of $X := \mathbb{R}$, which is Hausdorff.

(iii) False. Again the space Y in Problem 7.1 is not Hausdorff (hence not normal) and it is quotient of $X := \mathbb{R}$, which is normal.

(iv) False. For any topological space X, the constant map $f\colon X \to \{*\}$ is a continuous surjection, where $\{*\}$ is a topological space with a single element. Moreover, it is trivial to check that $\{*\}$ is equipped with the quotient topology. However, this contradicts the statement if X is a space with infinitely many points.

(v) True, because Y is the continuous image of a connected set.

(vi) False. Since metric spaces are Hausdorff, the space in Problem 7.1 furnishes a counterexample.

Solution of 7.4: Let C be a connected component of Y. We claim that $q^{-1}(C)$ is the union of connected components of X (in general we cannot expect $q^{-1}(C)$ to be a single connected component). Consider a connected component K of X such that $q(K) \cap C \neq \emptyset$, we want to prove that $q(K) \subseteq C$. Observe that $q(K)$ is connected, thus $q(K) \cup C$ is a connected set that contains C. As a result, since C is a connected component, $q(K) \cup C$ must coincide with C, that is $q(K) \subseteq C$, as we wanted. Thus $q^{-1}(C) = \bigcup_{i \in I} K_i$, where K_i is a connected component of X for every $i \in I$. Since, by assumption, K_i is open for all $i \in I$, we obtain that $q^{-1}(C)$ is open. Hence, by the definition of quotient topology, we conclude that C is open in Y.

Solution of 7.5: We start by showing that $\mathrm{int}(A)$ is saturated. Since A is saturated we have $\mathrm{int}(A) \subseteq q^{-1}(q(\mathrm{int}(A))) \subseteq q^{-1}(q(A)) = A$. However, since q is open, we have that $q(\mathrm{int}(A))$ is open. Therefore $q^{-1}(q(\mathrm{int}(A)))$ is open, because q is continuous. As a result, since $\mathrm{int}(A)$ is the largest open that is contained in A, we get $\mathrm{int}(A) = q^{-1}(q(\mathrm{int}(A)))$.

For the closure, observe that the fact that A is saturated implies that $X \setminus A$ is saturated. Thus $\mathrm{int}(X \setminus A)$ is saturated, which implies that $X \setminus \mathrm{int}(X \setminus A) = \overline{A}$ is also saturated.

For the counterexample, consider the quotient of the interval $[-2, 2]$ by the equivalence relation $-1 \sim 1$. Given $A := (0, 1)$, we have that A is saturated, but $\overline{A} = [0, 1]$ is not. On the other hand, defining $B := [-2, 0] \cup [1, 2]$, we have that B is saturated, but $\mathrm{int}(B) = [-2, 0) \cup (1, 2]$ is not.

Solution of 7.6: Let us consider the map

$$\begin{aligned} f\colon [0,1] \times [0,1] &\to S^1 \times S^1 \subseteq \mathbb{R}^4 \\ (s,t) &\mapsto (\cos(2\pi s), \sin(2\pi s), \cos(2\pi t), \sin(2\pi t)), \end{aligned}$$

which is easily a continuous map. Moreover observe that $f(s, 0) = (\cos(2\pi s), \sin(2\pi s), 1, 0) = f(s, 1)$ for all $s \in [0, 1]$ and similarly $f(0, t) = f(1, t)$ for all $t \in [0, 1]$. Hence the map f descends to the quotient $Q/\!\sim$ as a continuous map $\tilde{f}\colon X_2 \to S^1 \times S^1$ (here we have employed the universal property of topological quotients). We want to show that $\tilde{f}$ is a homeomorphism. It is straightforward to check that $\tilde{f}$ is bijective, hence the result follows from the *homeomorphism criterion* (seen in class as Proposition 2 of Lecture 9, see statement below), since X_2 is compact (being the quotient of a compact space, hence the image of a compact space via the quotient map) and $S^1 \times S^1$ is Hausdorff (being a topological subspace of $\mathbb{R}^4$ which is Hausdorff).

Proposition (Homeomorphism criterion)**.** *Let X, Y be topological spaces and let $f : X \to Y$ be a continuous bijection. If X is compact and Y is Hausdorff, then f is a homeomorphism.*

Solution of 7.7: We first prove that X_1 is homeomorphic to X_2. Consider the function

$$\begin{aligned} f\colon D^2 \subseteq \mathbb{R}^2 &\to S^2 \subseteq \mathbb{R}^3 \\ (x,y) &\mapsto (x, y, \sqrt{1-(x^2+y^2)}). \end{aligned}$$

Observe that f maps homeomorphically D^2 to the northern hemisphere $\{(x, y, z) \in S^2 : z \geq 0\}$ of S^2. Denote by $p\colon S^2 \to S^2/\!\sim\, = X_1$ and $q\colon D^2 \to D^2/\!\simeq\, = X_2$ the quotient maps. Note that $p \circ f(u) = p \circ f(v)$ for all $u \simeq v$ in D^2. Hence g descends to the quotient as a continuous map $g\colon X_2 \to X_1$, i.e. the diagram below commutes.

$$\begin{array}{ccc} D^2 & \xrightarrow{f} & S^2 \\ \downarrow{\scriptstyle q} & & \downarrow{\scriptstyle p} \\ X_2 & \xrightarrow{g} & X_1 \end{array}$$

It is really easy to check that the map g is bijective, hence by the *homeomorphism criterion* (see in proof of Problem 7.6 or Proposition 2 in Lecture 9) we obtain that g is a homeomorphism as we wanted. Indeed X_2 is compact (being the quotient of the compact space D^2) and X_1 is Hausdorff. To see that X_1 is Hausdorff, given any $u, v \in S^2$ with $p(u) \neq p(v)$, we can find open neighborhoods $U, V \subseteq S^2$ of u, v respectively that are symmetric with respect to the origin (i.e. $U = -U$ and $V = -V$) and such that $U \cap V = \emptyset$. Then $p(U)$, $p(V)$ are disjoint open neighborhoods of $p(u)$ and $p(v)$.

We now prove that X_1 is homeomorphic to X_3. Consider the map $h\colon S^2 \subseteq \mathbb{R}^3 \to X_3$ such that $h(u)$ is defined as the line passing through u and the origin in $\mathbb{R}^3$, for all $u \in S^2$. It is straightforward to check that the map h is continuous and surjective. Moreover $h(u) = h(v)$ if and only if $u = v$ or $u = -v$. Hence, h descends to the quotient as a bijective continuous map $\tilde{h}\colon X_1 \to X_3$. As a result, again by the *homeomorphism criterion* (X_1 is compact, being quotient of S^2, and X_3 is Hausdorff, being a metric space), we have that $\tilde{h}$ is a homeomorphism, which concludes the proof.

Solution of 7.8: Denote by $X = \mathbb{R}^n \cup \{\infty\}$ the Alexandroff one-point compactification of $\mathbb{R}^n$ and consider the stereographic projection $\varphi\colon S^n \subseteq \mathbb{R}^{n+1} \to X$ defined as

$$\varphi(x_1, \dots, x_{n+1}) := \begin{cases} \left(\frac{x_1}{1-x_{n+1}}, \dots, \frac{x_n}{1-x_{n+1}}\right) & \text{if } (x_1, \dots, x_{n+1}) \neq N \\ \infty & \text{if } (x_1, \dots, x_{n+1}) = N, \end{cases}$$

where $N = (0, \dots, 0, 1)$ is the north pole of the sphere. Note that f is easily bijective. Moreover φ is continuous. Indeed consider any open set $U \subseteq X$. If $U \subseteq \mathbb{R}^n$, then $\varphi^{-1}(U)$ is open in S^n since the standard stereographic projection $\varphi\colon S^n \setminus \{N\} \to \mathbb{R}^n$ is a homeomorphism. If $\infty \in U$, then $U = X \setminus C$ where C is a compact subset of $\mathbb{R}^n$ (by definition of Alexandroff one-point compactification). Hence $\varphi^{-1}(C)$ is a compact subset of $S^n \setminus \{N\}$ (again because the standard stereographic projection is a homeomorphism), which implies that $\varphi^{-1}(U) = S^n \setminus \varphi^{-1}(C)$ is open in S^n. Therefore $\varphi\colon S^n \to X$ is a bijective continuous map, thus it is a homeomorphism by the lemma in the *homeomorphism criterion* (S^n is compact and X is Hausdorff, both easy to check).

Solution of 7.9:

(i) Assume that X is Hausdorff, we first show that $Y = (X \times X) \setminus \Delta$ is open. Let $(x, y) \in Y$, that is $x \neq y$. Then there exists disjoint open sets $U, V \subseteq X$ such that $x \in U$ and $y \in V$. Then $U \times V$ is an open subset of $X \times X$ that contains (x, y). We claim that is contained in Y. Indeed, suppose that there were $z \in X$ such that $(z, z) \in U \times V$, then $z \in U \cap V$, which is a contradiction.

Viceversa, assume that the space Y above is open, and take any distinct points $x, y \in X$, i.e. $(x, y) \in Y$. Since Y is open, there is an open set O contained in Y that contains (x, y). By the definition of product topology, we can write O as $O = \bigcup_{i\in I} U_i \times V_i$, where U_i and V_i are open subsets of X for all $i \in I$. In particular, there exists $j \in I$ such that $(x, y) \in U_j \times V_j \subseteq O \subseteq Y$. Hence, U_j, V_j are disjoint open sets containing x, y respectively, which is sufficient to prove that X is Hausdorff.

(ii) First, assume that $X/\sim$ is Hausdorff, and let $(x, y) \in (X \times X) \setminus R$. By definition of R we have that $q(x) \neq q(y)$. Thus we can find disjoint open sets U_x, U_y in $X/\sim$ that contain $q(x)$ and $q(y)$ respectively. By definition of quotient topology, $q^{-1}(U_x)$ and $q^{-1}(U_y)$ are open in X. Moreover, $q^{-1}(U_x) \times q^{-1}(U_y) \subseteq (X \times X) \setminus R$. Indeed, suppose that there were a point $(u, v) \in (q^{-1}(U_x) \times q^{-1}(U_y)) \cap R$. Then $u \sim v$ which contradicts U_x and U_y being disjoint in $X/\sim$. Therefore for every $(x, y) \in (X \times X) \setminus R$ we found an open neighborhood contained in $(X \times X) \setminus R$, hence $(X \times X) \setminus R$ is open and R is closed.

Viceversa, assume that R is closed. By Problem 2.4, we have that $q \times q\colon X \times X \to (X/\sim) \times (X/\sim)$ is open. Hence, since $(X \times X) \setminus R$ is open by hypothesis, we obtain that $q((X \times X) \setminus R)$ is open in $(X/\sim) \times (X/\sim)$. However observe that $q((X \times X) \setminus R) = ((X/\sim) \times (X/\sim)) \setminus \Delta_\sim$, where $\Delta_\sim$ is the diagonal of $X/\sim$. Hence, $\Delta_\sim$ is closed in $(X/\sim) \times (X/\sim)$ and, by (i), this proves that $X/\sim$ is Hausdorff.

Solution of 7.10:

(i) Let $q\colon X \to X/K$ be the quotient map. The elements of X/K are equivalence classes $[x]$ such that $[x] = \{x\}$ if $x \notin K$ and $[x] = K$ otherwise. Let $[x] \neq [y]$ be distinct points of X/K. There are two cases: either $x, y \in X \setminus K$, or (up to exchanging x and y) $x \in X \setminus K$ and $[y] = K$. In the first case, since X is Hausdorff there exist two disjoint open sets U, V such that $x \in U$ and $y \in V$. Replacing U and V with $U \cap (X \setminus K)$ and $V \cap (X \setminus K)$ respectively (recall that, since X is Hausdorff, K is closed and $X \setminus K$ is open), we can assume that U, V do not intersect K. We claim that $q(U)$ and $q(V)$ are disjoint open sets in X/K that contain $[x]$ and $[y]$ respectively. It is clear that $[x] \in q(U)$, $[y] \in q(U)$ and that $q(U) \cap q(V) = \emptyset$. Moreover, by the definition of the quotient map, $q^{-1}(q(U)) = U$ and similarly for V. Thus, by definition of quotient topology, $q(U)$ and $q(V)$ are open.

Consider now the case $x \in X \setminus K$ and $[y] = K$. If we can find disjoint open sets U and V such that $x \in U$ and $K \subseteq V$, then the same reasoning as before gives that $q(U)$ and $q(V)$ are disjoint open sets containing $[x], [y]$ respectively. For each point $k \in K$, let U_k and V_k be disjoint open sets such that $x \in U_k$ and $k \in V_k$. The existence of such open sets is guaranteed by the fact that X is Hausdorff. Note that $\{V_k \cap K\}_{k\in K}$ is an open cover for K, thus it admits a finite subcover $\{V_h\}_{h\in H}$. Then set $U := \bigcap_{h\in H} U_h$, $V := \bigcup_{h\in H} V_h$. Since H is finite, both U and V are open. Moreover, it is clear by construction that U and V are disjoint and contain respectively x and K, which concludes the proof.

(ii) First, let us introduce some notation. Let $Z := (X \setminus A)/(K \setminus A)$ and $Y = X/K$. Let $i\colon X \setminus A \to X$ be the inclusion map, and let $q\colon X \to Y$ and $p\colon X \setminus A \to Z$ be the quotient maps.

$$\begin{array}{ccc} X \setminus A & \xhookrightarrow{\quad i \quad} & X \\ \big\downarrow p & & \big\downarrow q \\ Z := (X \setminus A)/(K \setminus A) & \xrightarrow{\ f\ } & Y := X/K \end{array}$$

First we show that f is well-defined. Consider two points $x, y \in X \setminus A$ such that $[x]_A = [y]_A$. This means that $x = y$ or $x, y \in K \setminus A$. In the first case, obviously we have $[x] = [y]$ in X/K. In the second case, in particular we have $x, y \in K$, hence again $[x] = [y]$ in X/K. Thus the function f is well-defined.

We now prove that f is a homeomorphism. We subdivide the proof in different steps.

Step 1. f is injective. Consider two points $x, y \in X \setminus A$ such that $[x] = [y]$ in X/K. Then $x = y$ or $x, y \in K$, so $x, y \in K \setminus A$. In both cases we have $[x]_A = [y]_A$. Hence f is injective.

Step 2. f is surjective. Consider a point $x \in X$. If $x \in X \setminus K$, then $x \in X \setminus A$. Thus $[x]_A$ is well-defined and $[x] = f([x]_A)$. Otherwise assume $x \in K$. Since $A \subsetneq K$, there is $y \in K \setminus A \subseteq X \setminus A$. Therefore $[y]_A$ is well-defined and we have $[x] = [y] = f([y]_A)$, where the first equality follows from the fact that $x, y \in K$.

Step 3. f is continuous. By the universal property of the quotient, the continuity of f is equivalent to the continuity of $f \circ p \colon X \to X/K$. By the definition of f, we have that the diagram above commutes, that is $q \circ i = f \circ p$. Since q and i are continuous, so it is $q \circ i$. Thus f is continuous.

Step 4. f is open. Let U be an open set in Z, we want to prove that $f(U)$ is open. Observe that U is open if and only if $p^{-1}(U)$ is open in $X \setminus A$. Now we distiguish two cases:

- U does not contain the equivalence class relative to points in $K \setminus A$, i.e. $p^{-1}(U) \subseteq X \setminus K$. In this case, note that $X \setminus K$ is open (since X is Hausdorff and K is compact), hence $p^{-1}(U)$ is open seen as a subset of X. Therefore $i \circ p^{-1}(U)$ is open and contained $X \setminus K$. As a result, since $q^{-1}(q(i \circ p^{-1}(U))) = i \circ p^{-1}(U)$ (i.e. $i \circ p^{-1}(U)$ is saturated in the notation of Problem 7.5), $q(i \circ p^{-1}(U))$ is open in X/K, which concludes that $f(U)$ is open since $f(U) = q(i \circ p^{-1}(U))$.
- U contains the equivalent class relative to points in $K \setminus A$, i.e. $p^{-1}(U)$ contains $K \setminus A$. Then observe that $i \circ p^{-1}(U) \cup A$ is open seen as a subset of X, since A is open. Moreover $i \circ p^{-1}(U) \cup A$ contains K. Hence $i \circ p^{-1}(U) \cup A$ is a again saturated set in X with respect to the quotient $q \colon X \to X/K$. Hence, $q(i \circ p^{-1}(U) \cup A)$ is open in X/K. Moreover observe that $q(i \circ p^{-1}(U) \cup A) = f(U)$, which proves what we want.

From this four steps it follows immediately that f is a homeomorphism as we wanted.

(iii) Assume that $A = K$. Then $(X \setminus A)/(K \setminus A) = X \setminus K$, which in general is not the same as X/K. This can be easily seen if X is the space $\{a, b\}$ equipped with the discrete topology, and $A = K = \{b\}$. Then $X \setminus K = \{a\}$ while $X/K = \{[a], [b]\}$ has two elements.

(iv) Let $i\colon X \setminus K \hookrightarrow X$ be the standard inclusion and let $W := (X \setminus K) \cup \{\infty\}$ be the Alexandroff one-point compactification of $X \setminus K$. Then consider the map $g\colon X \to W$ defined as

$$g(x) = \begin{cases} i^{-1}(x) & \text{if } x \notin K \\ \infty & \text{if } x \in K. \end{cases}$$

This map is easily surjective and $g(x) = g(y)$ if and only if $x = y$ or $x, y \in K$. Let us prove that g is continuous. Take any open set $U \subseteq W$. If $U \subseteq X \setminus K$, then $g^{-1}(U) = i^{-1}(U)$, which is open since $i^{-1}(U)$ is open in $X \setminus K$ and $X \setminus K$ is open in X (recall that K is compact, hence closed by the Hausdorff assumption on X). If $\infty \in U$, then $U = W \setminus C$ where C is a compact subset of $X \setminus K$. Hence $g^{-1}(U) = X \setminus i^{-1}(C)$, which is open since $i^{-1}(C)$ is compact and thus closed (again by the Hausdorff assumption on X).

As a result, g descends to the quotient as a bijective continuous map $\tilde{g}\colon X/K \to W$. Let us prove that $\tilde{g}$ is also open, which implies that $\tilde{g}$ is a homeomorphism as we wanted. Consider any open subset $U \subseteq X/K$, then $p^{-1}(U)$ is open in K. If $p^{-1}(U) \subseteq X \setminus K$, we get that $\tilde{g}(U) = g(p^{-1}(U)) = i^{-1} \circ p^{-1}(U)$ is open in W. On the other hand, if $p^{-1}(U) \supseteq K$, then

$$\tilde{g}(U) = g(p^{-1}(U)) = W \setminus g(X \setminus p^{-1}(U)) = W \setminus i^{-1}(X \setminus p^{-1}(U)).$$

Note that $X \setminus p^{-1}(U)$ is compact in X, since $p^{-1}(U)$ is open and X is compact. Hence, $i^{-1}(X \setminus p^{-1}(U))$ is compact in $X \setminus K$ and thus $\tilde{g}(U) = W \setminus i^{-1}(X \setminus p^{-1}(U))$ is open in W by definition of Alexandroff one-point compactification.