8. Homotopy and contractible spaces

Chef's table

In this problem set, we start working with the most basic notion in Algebraic Topology, that of *homotopy* (of maps, and topological spaces). The first seven exercises deal, one way or the other, with contractible spaces (those spaces that are equivalent to a point, so the *simplest ones* we can think of). They are all rather simple, except for 8.5 which is a bit more tedious, and also partly unrelated to the other ones (so maybe skip it at a first reading, and get back to it later). Through Problems 8.6 and 8.7 we introduce a basic operation (called *cone*), which produces a topological space $C(X)$ given a topological space X. As it often happens in Mathematics, the name is not random: if you take $X = S^1$ then $C(X)$ is an honest ice-cream cone.

Problem 8.8 is a first (and most important!) example of 'non-trivial' homotopy between maps on spheres, and it introduces some ideas that will come back later in my lectures, when I will prove tha so-called *hairy ball theorem*.

Lastly, Problem 8.9 and Problem 8.10 are meant to clearly suggest the difference between homotopies (of loops) fixing the basepoint and homotopies which are allowed to move basepoints around. The two notions are *heavily inequivalent* and these two exercises should guide you to build a space which exhibits this inequivalence. These outcome is conceptually very important, and (even if you can't solve 8.10) the result should be known and understood by all of you.

8.1. Contractible spaces are simply connected ⚙. Let X be a contractible space.

(i) Show that X is path-connected.

(ii) Prove that $\pi_1(X) \cong \{1\}$.

Note: Point (ii) *of this exercise follows directly from the homotopy invariance of the fundamental group. However, in this special case, the proof is much simpler.*

8.2. Functions on contractible spaces ✎. Prove the following statements.

(i) Let X be a path-connected topological space. Show X is contractible if and only if for any path-connected topological space Y and any pair of functions $f, g\colon X \to Y$, we have that f and g are homotopic.

(ii) Show that a topological space X is contractible if and only if for any topological space Y and any pair of continuous function $f, g\colon Y \to X$, we have that f and g are homotopic.

8.3. Homotopic paths ✍. Let X be a topological space, and let γ_1, γ_2 be paths in X with the same endpoints (i.e. $\gamma_1(0) = \gamma_2(0)$ and $\gamma_1(1) = \gamma_2(1)$). Show that γ_1 and γ_2 are homotopic if and only if there is a continuous map $F\colon D^2 \to X$ such that $F|_{\partial D^2}\colon S^1 \to X$ is a reparametrization of $\gamma_1 * \gamma_2^{-1}$.

8.4. Trivial fundamental group ✍. Let X be a path-connected topological space. Show that $\pi_1(X) = \{1\}$ if and only if for every pair of points x, y of X, there exists only one homotopy class of paths joining them.

8.5. Path of continuous functions ⚙. Let $I = [0,1]$, and let (X, d) be a metric space. Let $\gamma_0, \gamma_1\colon I \to X$ be paths in X with the same endpoints. Let $\mathcal{S}$ be the subset of $C(I, X)$ defined as $\mathcal{S} := \{f \in C(I,X) \,:\, f(0) = \gamma_0(0) = \gamma_1(0), f(1) = \gamma_0(1) = \gamma_1(1)\}$. Show that there is a continuous path $\Gamma\colon [0,1] \to \mathcal{S}$ between γ_0 and γ_1 (i.e. $\Gamma(0) = \gamma_0$, $\Gamma(1) = \gamma_1$) if and only if γ_0 and γ_1 are homotopic.

Note: Recall that $C(I, X) = \{f\colon I \to X \,:\, f$ is continuous$\}$ is a metric space if equipped with the distance $d(f,g) = \sup_{t\in[0,1]}\{d_X(f(t), g(t))\}$ (see Problem 6.1).

8.6. Cone that is not a topological manifold ✍. Find an example of a topological manifold X such that the cone $C(X) := (X \times [0,1])/(X \times \{0\})$ over X is not a topological manifold around $\bar{x} = [X \times \{0\}] \in C(X)$, i.e. the point $\bar{x}$ does not admit any neighborhood $U \subseteq C(X)$ of $\bar{x}$ homeomorphic to $\mathbb{R}^n$ for some $n \in \mathbb{N}$.

8.7. Every cone is contractible ⚙. Given a topological space X, let $C(X)$ denote the cone over X, i.e. $C(X) := (X \times [0,1])/(X \times \{0\})$. Show that

(i) $C(X)$ is path-connected;

(ii) $C(X)$ is contractible.

8.8. Homotopic maps on the sphere ⚙. Show that, if $n \in \mathbb{N}$ is odd, the antipodal map $-\mathrm{Id}_{S^n}\colon S^n \to S^n$ on the sphere is homotopic to the identity map Id_{S^n}.

8.9. Preparation to Problem 8.10 ⚙. Recall that a space X is contractible if there exists a continuous map $H\colon X \times [0,1] \to X$ and a point $x_0 \in X$ such that $H(x, 0) = x$ and $H(x, 1) = x_0$ for every $x \in X$. Note that we do not require that $H(x_0, t) = x_0$ for every possible $t \in [0,1]$. If we add this hypothesis, i.e. we ask $H(x_0, t) = x_0$ for all $t \in [0,1]$, then we say that X *deformation retracts* to a point (more precisely to the point x_0). The goal of this and the following exercises is to show that deformation retracting to a point is a stronger property than being contractible.

(i) Show that, if a space X deformation retracts to a point $x_0 \in X$, then for each neighborhood $U \subseteq X$ of x_0 there exists a neighborhood $V \subseteq U$ of x_0 such that the inclusion map $i\colon V \hookrightarrow U$ is homotopic in U to the constant map $c_{x_0}\colon V \to \{x_0\}$.

(ii) Let X be the subspace of $\mathbb{R}^2$ consisting of the horizontal segment $[0,1]\times\{0\}$ together with all the vertical segments $\{r\}\times[0,1-r]$, for r a rational number in $[0,1]$. Show that the space X deformation retracts to any $x_0 \in [0,1]\times\{0\}$, but not to any other point.

8.10. Contractible spaces that do not deformation retract. Construct an example of a topological space X that is contractible but does not deformation retract to any point $x_0 \in X$.

8. Solutions

Solution of 8.1: Let $H\colon X \times [0,1] \to X$ be a homotopy between the identity on X and a constant map, i.e. $H(x,0) = x$ and $H(x,1) = x_0$ for all $x \in X$, for some $x_0 \in X$. Recall that this map H exists since X is contractible. Point (i) is very simple. Indeed, given two points $y, z \in X$, we can define a continuous path $\gamma\colon [0,1] \to X$ between them as

$$\gamma(s) := \begin{cases} H(y, 2s) & \text{if } s \in [0, 1/2] \\ H(z, 2-2s) & \text{if } s \in [1/2, 1]\,. \end{cases}$$

Hence, let us prove point (ii). In particular, we will show that $\pi_1(X, x_0) \cong \{1\}$. Consider a loop $\gamma\colon [0,1] \to X$ based in x_0, i.e. $\gamma(0) = \gamma(1) = x_0$. Then, the map $F\colon [0,1]\times[0,1] \to X$, defined as

$$F(s,t) := H(\gamma(s), t),$$

is such that $F(s,0) = \gamma(s)$, $F(s,1) = x_0$. Note that the curve $\alpha\colon [0,1] \to X$ given by $\alpha(t) := F(0,t) = H(x_0, t)$ is a continuous path with $\alpha(0) = \alpha(1) = x_0$. In particular, α is an element of $\pi_1(X, x_0)$.

Now consider the homotopy map $G\colon [0,1] \times [0,1] \to X$ given by $G(s,t) := H(\alpha(s), t)$ and define $K\colon [0,1] \times [0,1] \to X$ as (see Figure 1 for a representation of the definition)

$$K(s,t) = \begin{cases} G(t, 1-3s) & \text{if } s \in [0, 1/3] \\ F(3s-1, t) & \text{if } s \in [1/3, 2/3] \\ G(t, 3s-2) & \text{if } s \in [2/3, 1]\,. \end{cases}$$

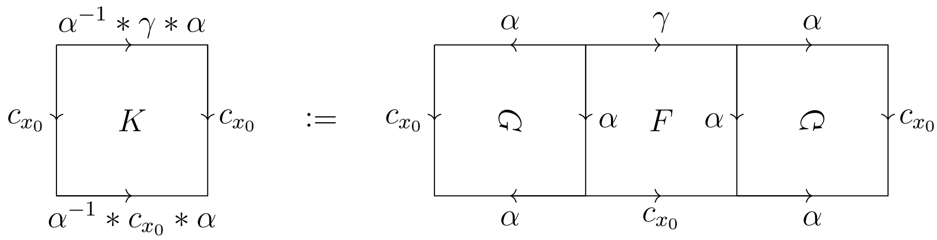

Figure 1: Definition of K.

It is not difficult to check that K is continuous, $K(\cdot, 0) = \alpha^{-1} * \gamma * \alpha$, $K(\cdot, 1) = \alpha^{-1} * c_{x_0} * \alpha$ and $K(0,t) = K(1,t) = x_0$ for all $t \in [0,1]$. As a result, K is a homotopy between $\alpha^{-1}*\gamma*\alpha$ and $\alpha^{-1} * c_{x_0} * \alpha$, from which we obtain that

$$[\alpha^{-1} * \gamma * \alpha] = [\alpha^{-1} * c_{x_0} * \alpha] \implies [\gamma] = [\alpha] \cdot [\alpha^{-1} * c_{x_0} * \alpha][\alpha^{-1}] = [c_{x_0}]$$

Hence $\pi_1(X, x_0) = \{[c_{x_0}]\} = \{1\}$, as we wanted.

Solution of 8.2: We prove the two statements separately.

(i) First assume that X is contractible and let $H\colon X\times[0,1]\to X$ be a homotopy between the identity on X and a constant map, i.e. $H(x,0)=x$ and $H(x,1)=x_0$ for every $x\in X$, for some $x_0\in X$. Now define $y_1:=f(x_0)$ and $y_2:=g(x_0)$. Finally, let $\gamma\colon[0,1]\to Y$ be a path in Y between y_1 and y_2, which exists by path-connectedness of Y. Consider the map $F\colon X\times[0,1]\to Y$ defined as

$$F(x,s):=\begin{cases} f(H(x,3s)) & \text{if } s\in[0,1/3]\\ \gamma(3s-1) & \text{it } s\in[1/3,2/3]\\ g(H(x,3-3s)) & \text{it } s\in[2/3,1]\,. \end{cases}$$

It is clear that F is continuous and that $F(x,0)=f(x)$, $F(x,1)=g(x)$ for all $x\in X$. In particular, F is an homotopy between f and g.

Viceversa, suppose that any pair of continuous functions $f,g:X\to Y$ are homotopic, for any path-connected topological space Y. Consider the case $Y=X$, $f=\mathrm{Id}_X$ and $g=c_{x_0}$ for some $x_0\in X$. Then we get that there is a homotopy $H\colon X\times[0,1]\to X$ such that $H(x,0)=x$ and $H(x,1)=c_{x_0}(x)=x_0$ for every $x\in X$. Thus H is the desired homotopy between the identity on X and a constant map.

(ii) Suppose that X is contractible, and let $H\colon X\times[0,1]\to X$ be a homotopy between the identity and a constant map. Let $F\colon Y\times[0,1]\to X$ be the map defined as:

$$F(y,s)=\begin{cases} H(f(y),2s) & \text{if } s\in[0,1/2]\\ H(g(y),2s-1) & \text{it } s\in[1/2,1]\,. \end{cases}$$

It is clear that F is continuous and that $F(y,0)=f(y), F(y,1)=g(y)$ for all $y\in Y$. In particular, F is a homotopy between f and g.

Vicerversa, suppose that, given a topological space Y, any pair of functions $f,g:Y\to X$ are homotopic. As before, consider the case $Y=X$, $f=\mathrm{Id}_X$ and $g=c_{x_0}$ for some $x_0\in X$. Then there is a homotopy $H\colon X\times[0,1]\to X$ such that $H(x,0)=x$ for every $x\in X$ and $H(x,1)=c_{x_0}(x)=x_0$. Thus H is the desired homotopy showing that X is contractible.

Solution of 8.3: Suppose that γ_1 and γ_2 are homotopic. This means that there exists a continuous map $H\colon[0,1]\times[0,1]\to X$ such that $H(0,t)=\gamma_1(t)$ and $H(1,t)=\gamma_2(t)$ for all $t\in[0,1]$; and moreover $H(s,0)=\gamma_1(0)=\gamma_2(0)$ and $H(s,1)=\gamma_1(1)=\gamma_2(1)$ for all $s\in[0,1]$. We know that $[0,1]\times[0,1]$ is homeomorphic to the disk D^2, let $f\colon D^2\to[0,1]\times[0,1]$ be such the homeomorphism given in the solution of point (iii) in Problem 2.1. Then $F:=H\circ f\colon D^2\to X$ is a continuous map. It is not hard to see that $F((1-s)\pi,1)$ is a reparametrization of γ_1, and similarly $F((1+s)\pi,1)$ is a reparametrization of γ_2, as we wanted.

For the second part, let $F\colon D^2\to X$ be as in the hypotheses. Without loss of generality we can assume that $\gamma_1(s)=F(\cos((1-s)\pi),\sin((1-s)\pi))$ and $\gamma_2(s)=F(\cos((1+s)\pi),\sin((1+s)\pi))$ for all $s\in[0,1]$, where the disk D^2 is parametrized in Euclidean coordinates. We can explicitly define a homotopy $H\colon[0,1]\times[0,1]\to X$ between γ_1 and γ_2 as

$$H(s,t):=F(\cos((1-s)\,\pi),(1-2t)\sin((1-s)\,\pi)).$$

Solution of 8.4: Assume that $\pi_1(X) = \{1\}$ and take two points $x, y \in X$ and paths γ_1 and γ_2 joining them. Consider the juxtaposition $\gamma = \gamma_1 * \gamma_2^{-1}$. Since $\pi_1(X, x) \cong \pi_1(X) = \{1\}$ and γ is a loop based at x, we have that γ is homotopic to the trivial path on x (i.e. the path $c_x(t) = x$ for all $t \in [0, 1]$). Thanks to Problem 8.3, the image of $\gamma * c_x^{-1} = \gamma * c_x$ bounds a disk. However note that $\gamma * c_x$ is just a reparametrization of $\gamma = \gamma_1 * \gamma_2^{-1}$. As a result, again by Problem 8.3, γ_1 and γ_2 are homotopic. Hence, by arbitrariness of γ_1, γ_2, there is only one homotopy class of paths joining x_1, x_2.

Viceversa, suppose that for every pair of points $x, y \in X$, we have that there is only one homotopy class of paths joining them. Let $x \in X$ be any point. Then there exists only one homotopy class of paths between x and x. Thus, all loops based at x are homotopic, which means that $\pi_1(X, x) = \{1\}$.

Solution of 8.5: Let $\Gamma \colon I \to \mathcal{S}$ be a path between γ_0 and γ_1. We define a homotopy H between γ_0 and γ_1 as $H(s, t) = \Gamma(t)(s)$. Since Γ is a path contained in $\mathcal{S}$, we only need to show that H is continuous. Pick any $\varepsilon > 0$ and any point $(s, t) \in [0, 1] \times [0, 1]$. We want to show that there exists $\delta > 0$ such that if $|s-s'|+|t-t'| < \delta$, then $d_X(H(s, t), H(s', t')) < \varepsilon$. By triangular inequality, it holds that

$$\begin{aligned} d_X(H(s,t), H(s',t')) &\le d_X(H(s,t), H(s,t')) + d_X(H(s,t'), H(s',t')) \\ &= d_X(\Gamma(s)(t), \Gamma(s)(t')) + d_X(\Gamma(s)(t'), \Gamma(s')(t'). \end{aligned}$$

Since $\Gamma(s) \in C(I, X)$ (i.e. it is a continuous function from I to X), there exists $\delta_1 > 0$ such that if $|t - t'| < \delta_1$, then $d(\Gamma(s)(t), \Gamma(s)(t')) \le \varepsilon/2$. Moreover, since Γ is continuous, we have that there exists $\delta_2 > 0$ such that if $|s - s'| < \delta_2$, then $d_X(\Gamma(s)(t'), \Gamma(s')(t')) \le d_{C(I,X)}(\Gamma(s), \Gamma(s')) \le \varepsilon/2$. As a result, set $\delta := \min\{\delta_1, \delta_2\}$, if $|s - s'| + |t - t'| < \delta$, then $|s - s'| < \delta \le \delta_2$ and $|t - t'| < \delta \le \delta_1$ and, by the triangular inequality above, we get the result.

Now, suppose that γ_0 and γ_1 are homotopic via a map $H \colon [0, 1] \times [0, 1] \to X$, and let $H_t \colon I \to X$ be the map defined as $H_t(s) := H(s, t)$. Consider the map $\Gamma \colon I \to C(I, X)$ defined as $\Gamma(t) := H_t$. It is clear that $\Gamma(0) = \gamma_0$ and $\Gamma(1) = \gamma_1$. We claim that Γ is continuous. For every $\varepsilon > 0$ and $t \in [0, 1]$, we need to find a $\delta > 0$ such that if $|t - t'| < \delta$, then $d_{C(I,X)}(\Gamma(t), \Gamma(t')) \le \varepsilon$. Since $[0, 1] \times [0, 1]$ is compact and H is continuous, then H is uniformly continuous. In particular, there exists $\delta_1 > 0$ such that if $|s - s'| + |t - t'| < \delta_1$, then $d_X(H(s, t), H(s', t')) < \varepsilon$. Fix $\delta = \delta_1/2$ and let $t' \in [0, 1]$ be such that $|t - t'| \le \delta$. Then

$$d_{C(I,X)}(\Gamma(t), \Gamma(t')) = \sup_{s\in I}\{d_X(\Gamma(t)(s), \Gamma(t')(s))\} = \sup_{s\in I}\{d_X(H(s,t), H(s,t'))\} < \varepsilon.$$

The fact that the image of Γ is contained in $\mathcal{S}$ is clear because H is a homotopy of paths.

Solution of 8.6: Consider a space $X = \{a, b, c\}$ with three elements equipped with the discrete topology. Then

$$C(X) = {(\{a\} \times [0,1]) \cup (\{b\} \times [0,1]) \cup (\{c\} \times [0,1])}\big/{\{(a,0),(b,0),(c,0)\}}$$

is the union of three closed segments where one of the end points of each segments is identified in the point $\bar{x} = [(a, 0)] = [(b, 0)] = [(c, 0)] \in C(X)$. Now consider any open neighborhood $U \subseteq C(X)$ of $\bar{x}$, then $U \setminus \bar{x}$ consists of three connected components (one for each segment). Hence U cannot be homeomorphic to $\mathbb{R}^n$ for any $n \in \mathbb{N}$. Indeed $\mathbb{R}^n$ without a point is connected for $n \geq 2$ and has two connected components for $n = 1$.

Solution of 8.7: We directly show that $C(X)$ is contractible, from which follows also that $C(X)$ is path-connected by Problem 8.1. Consider the homotopy $H\colon (X \times [0, 1]) \times [0, 1] \to (X \times [0, 1])$ defined as

$$H((x, t), s) := (x, (1 - s)t).$$

Note that $H((x, 0), s) = (x, 0)$ for all $x \in X$ and $s \in [0, 1]$. Hence, H generates a well-defined homotopy $\tilde{H}\colon C(X)\times[0, 1] \to C(X)$ as $\tilde{H}([(x, t)], s) = [H((x, t), s)] = [(x, (1-s)t)]$. Observe that $\tilde{H}([(x, t)], 0) = [(x, t)]$ for all $[(x, t)] \in C(X)$ and that $\tilde{H}([(x, t)], 1) = [(x, 0)]$ is constant for all $[(x, t)] \in C(X)$. Hence H is a homotopy between the identity on $C(X)$ and a constant map, as we wanted.

In order to say that $\tilde{H}$ is a homotopy and in particular that it is continuous, we implicitely used that

$$(p \times \mathrm{id})\colon (X \times [0, 1]) \times [0, 1] \to C(X) \times [0, 1]$$

is a quotient map, where $p\colon X \times [0, 1] \to C(X) = (X \times [0, 1])/(X \times \{0\})$ is the quotient map defining the cone $C(X)$. This is a consequence of the following lemma, which we prove for completeness.

Lemma. *Let $p\colon X \to Y$ be a quotient map, then*

$$(p \times \mathrm{id})\colon X \times [0, 1] \to Y \times [0, 1]$$

is a quotient map as well.

Note: The result is also true if we substitute the interval $[0, 1]$ with any locally compact Hausdorff space (the proof works almost the same).

Proof. We have to check that $(p \times \mathrm{id})$ is surjective (which is obvious) and that a subset $U \subseteq Y \times [0, 1]$ is open if and only if $(p \times \mathrm{id})^{-1}(U) \subseteq X \times [0, 1]$ is open. The function $(p \times \mathrm{id})$ is continuous; thus, if $U \subseteq Y \times [0, 1]$ is open then $(p \times \mathrm{id})^{-1}(U) \subseteq X \times [0, 1]$ is open. Now consider a subset $U \subseteq Y \times [0, 1]$ such that $\tilde{U} := (p \times \mathrm{id})^{-1}(U) \subseteq X \times [0, 1]$ is open and take a point $(x, t) \in \tilde{U}$.

We prove that there exists a saturated (with respect to $p\colon X \to Y$) open neighborhood $V \subseteq X$ of x such that $V \times A$ is contained in $\tilde{U}$ for some open subset $A \subseteq [0, 1]$. First take an open neighborhood $V_1 \times A$ of (x, t) such that $V_1 \times \overline{A} \subseteq \tilde{U}$ (the existence follows easily from the definition of product topology and the structure of $[0, 1]$). Note that $p^{-1}(p(V_1)) \times \overline{A}$ is contained in $\tilde{U}$. Therefore, there exists an open neighborhood $V_2 \subseteq X$ of x such that $p^{-1}(p(V_1)) \times \overline{A} \subseteq V_2 \times \overline{A} \subseteq \tilde{U}$. This follows from the compactness of $\overline{A}$ (check it!). We repeat the procedure to construct an open neighborhood $V_3 \subseteq X$ of x such that $p^{-1}(p(V_2)) \times \overline{A} \subseteq V_3 \times \overline{A} \subseteq \tilde{U}$ and so on, obtaining a chain of sets $V_1 \subseteq V_2 \subseteq V_3 \subseteq \ldots$; then define $V := \bigcup_{k=1}^{\infty} V_k$. It is not difficult to check that V is a saturated open neighborhood of x such that $V \times A \subseteq \tilde{U}$.

Finally, we only need to verify that $p(V \times A)$ is an open neighborhood of $p(x,t)$ contained in U, which proves that U is open by arbitrariness of (x,t). However, observe that $p(V \times A) = p(V) \times A$ and that $p(V)$ is open in Y, since $p^{-1}(p(V)) = V$ is open and $p\colon X \to Y$ is a quotient map. As a result, $p(V \times A)$ is open in $Y \times [0,1]$, which concludes the proof. $\square$

Solution of 8.8: Since n is odd, we can write $n = 2k-1$. Denote by $\underline{x} = (x_1, y_1, \dots, x_k, y_k)$ the coordinates on $S^n \subseteq \mathbb{R}^{n+1} = \mathbb{R}^{2k}$ and consider the map $v\colon S^n \to S^n$ given by $v(x_1, y_1, \dots, x_k, y_k) := (y_1, -x_1, \dots, y_k, -x_k)$. Note that $\langle \underline{x}, v(\underline{x}) \rangle = 0$ for all $\underline{x} \in S^n$, where $\langle \cdot, \cdot \rangle$ is the standard scalar product in $\mathbb{R}^{n+1}$. Then we can define the homotopy $H\colon S^n \times [0,1] \to S^n$ as

$$H(\underline{x}, t) := \cos(\pi t)\underline{x} + \sin(\pi t) v(\underline{x}).$$

Note that H is well-defined, i.e. $H(\underline{x}, t) \in S^n$, since

$$\begin{aligned} \|H(\underline{x}, t)\|^2 &= \|\cos(\pi t)\underline{x} + \sin(\pi t) v(\underline{x})\|^2 \\ &= \|\cos(\pi t)\underline{x}\|^2 + \|\sin(\pi t) v(\underline{x})\|^2 + 2\langle \cos(\pi t)\underline{x}, \sin(\pi t) v(\underline{x}) \rangle \\ &= \cos^2(\pi t)\|\underline{x}\|^2 + \sin^2(\pi t)\|v(\underline{x})\|^2 + 2\cos(\pi t)\sin(\pi t)\langle \underline{x}, v(\underline{x}) \rangle \\ &= \cos^2(\pi t) + \sin^2(\pi t) = 1. \end{aligned}$$

Moreover H is trivially continuous and $H(\underline{x}, 0) = \underline{x}$, $H(\underline{x}, 1) = -\underline{x}$ for all $\underline{x} \in S^n$. Hence, H is a homotopy between the identity and the antipodal map, as we wanted.

Note: In the proof we used the existence of a map $v\colon S^n \to S^n$ such that $\langle \underline{x}, v(\underline{x}) \rangle = 0$ for all $\underline{x} \in S^n$. We refer to the map v as a unitary vector field on S^n (vector field because $v(\underline{x})$ is orthogonal to $\underline{x}$ for all $\underline{x} \in S^n$ and unitary because $\|v(\underline{x})\| = 1$). The existence of a unitary vector field is equivalent to the result in the exercise. In particular, in S^n for n even the result does not hold (see the hairy ball theorem).

Solution of 8.9:

(i) Let $H\colon X \times [0,1] \to X$ be a homotopy between the identity and the constant map c_{x_0} in X such that $H(x_0, t) = x_0$ for all $t \in [0,1]$, and let $U \subseteq X$ be a neighborhood of x_0. By definition of neighborhood, U contains an open set O that contains x_0 (recall that in general neighborhoods do not need to be open). Since H is continuous, we have that $H^{-1}(O) \subseteq X \times [0,1]$ is an open set that contains $\{x_0\} \times [0,1]$. Hence, by definition of product topology on $X \times [0,1]$, for every $t \in [0,1]$ we can find open subsets $V_t \subseteq X$ and $I_t \subseteq [0,1]$ such that $(x_0, t) \in V_t \times I_t$ and $V_t \times I_t \subseteq H^{-1}(O)$.

Now note that $\{x_0\} \times [0,1]$ with the induced topology is homeomorphic to $[0,1]$, in particular it is compact. Thus, we can find a finite set $\{t_1, \dots, t_n\} \subseteq [0,1]$ such that

$$\{x_0\} \times [0,1] \subseteq \bigcup_{i=1}^{n} (V_{t_i} \times I_{t_i}) \subseteq H^{-1}(O).$$

Consider $V := \bigcap_{i=1}^{n} V_{t_i} \subseteq X$, which is an open neighborhood of x_0, since it is finite intersection of open sets containing x_0. By construction, $V \times [0,1] \subseteq \bigcup_{i=1}^{n}(V_{t_i} \times I_{t_i}) \subseteq H^{-1}(O)$, which means that $H(V \times [0,1]) \subseteq O \subseteq U$. Therefore, $H\colon V \times [0,1] \to U$ provides the desired homotopy from the inclusion $V \hookrightarrow U$ to the constant map c_{x_0}.

(ii) First, let $x_0 = (x, 0)$ for some $x \in [0,1]$. Then we can explicitly define a deformation retraction of X to x_0 as

$$H((u,v),t) = \begin{cases} (u, (1-2t)v) & \text{if } t \in [0, 1/2]\,, \\ ((2t-1)x + (2-2t)u, 0) & \text{if } t \in [1/2, 1]\,. \end{cases}$$

Now consider any $x_0 = (x, y) \in X$ with $y \neq 0$. Let $U \subseteq X$ be the neighborhood of x_0 obtained intersecting X with the ball of radius $y/2$ around x_0. It is clear that U is not path-connected, since it is the union of countably many disjoint vertical segments. Now consider any neighborhood $V \subseteq U$ of x_0 and pick any $z = (x_z, y_z) \in V$ with $x_z \neq x$. Note that every path between z and x_0 in X exits U. Thus, the inclusion $V \hookrightarrow U$ cannot be homotoped via some $H\colon V \times [0,1] \to U$ to the constant c_{x_0} inside U, because otherwise the map $\gamma(t) = H(z,t)$ would be a path from z to x_0 inside U. In particular, X does not reformation retract onto x_0.

Solution of 8.10: