9. Coverings

Chef's table

This problem set is totally about *playing with covering maps.* There are some examples to work out (specifically problems 9.1-9.2-9.3) and a few basic facts to prove (problems 9.4-9.5-9.6-9.7-9.8). Among these exercises (which are meant to be quite fast to solve, and basically only build on the very definition of covering map) I would like to stress the importance of Problem 9.7, which gives us an excuse to introduce the definition of *discrete subset* in a topological space (have a look at the note below and make sure you can follow the remarks in there). The fact that any fiber of a covering map is indeed a discrete subset is crucially employed in the proof of the monodromy theorem (Lecture 19) and, in a special case, already in computing the fundamental group of S^1 (Lecture 18). Problem 9.9 is a little lemma which extends the analogous statement for paths (i.e. that 'paths are evenly covered'), and is used in the proof of existence of lifts for homotopies. Last but not least, the challenge problem of this week is more accessible than other ones and is in fact an *excellent* way of testing your understanding of the ideas we have introduced in the last two weeks. This time I suggest that you all give it a look and try to think about it, as it may really clarify what Algebraic Topology is about.

9.1. Cover the circle. Let $p\colon \mathbb{R} \to S^1$ be the map defined as

$$p(t) := (\cos(2\pi t), \sin(2\pi t)).$$

Show that p is a covering map (of infinite degree).

9.2. Cover the circle, reloaded. Prove that $p\colon S^1 \to S^1$ given by $p(z) := z^n$ is a degree n covering map for any $n \geq 1$.

Note: Here we regard $S^1 \subseteq \mathbb{C}$ as unit circle.

9.3. Cover the punctured plane. Prove that $p\colon \mathbb{C}^* \to \mathbb{C}^*$ given by $p(z) := z^n$ is a degree n covering map for any $n \geq 1$.

Note: Here we have denoted $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$.

9.4. Two criteria for covering maps. Let $\tilde{X}, X$ be topological spaces and let $p\colon \tilde{X} \to X$ be a continuous function. Then:

(i) p is a covering map if and only if there is a cover of X consisting of evenly covered open sets;

(ii) p is a covering map if and only if there is a basis of X consisting of evenly covered open sets.

Deduce that a covering map is open.

9.5. Restrictions of covering maps ✍. Consider a covering map $p\colon \tilde{X} \to X$ and a subspace $A \subseteq X$. Defining $\tilde{A} := p^{-1}(A)$, show that the restriction $p|_{\tilde{A}}\colon \tilde{A} \to A$ is a covering map.

9.6. Composition of covering maps ✍. Let X, Y, Z be topological spaces, and let $p\colon X \to Y$ and $q\colon Y \to Z$ be covering maps. Assume that $q^{-1}(z)$ is a finite set for all $z \in Z$. Show that $q \circ p\colon X \to Z$ is a covering map.

9.7. Fibers are discrete ⚙. Let $p\colon \tilde{X} \to X$ be a covering map, with X a Hausdorff space. Show that for any $x \in X$ the fiber $p^{-1}(x)$ is a discrete subset of $\tilde{X}$. Deduce that, if $\tilde{X}$ is compact and X is connected, then the covering in question has finite degree.

Note: We say that a subset A of a topological space X is discrete if for all $x \in X$ there exists a neighborhood $U = U(x)$ such that $(U \setminus \{x\}) \cap A = \emptyset$. Observe that:

- *If $A \subseteq X$ is a discrete subset, then the induced topology on A as a subspace of X is discrete.*
- *The converse of the previous assertion is false, for instance $A := \{1/n : n \in \mathbb{N}_*\}$ inherits the discrete topology as a subspace of $\mathbb{R}$ (the standard real line) but it is not a discrete subset.*

Note that, as a result of the first item above, a continuous map $f\colon Y \to X$ with Y connected, such that $f(Y) \subseteq A$ with A discrete is actually constant.

9.8. When the fibers are finite ⚙. Let $p\colon \tilde{X} \to X$ be a covering map with $p^{-1}(x)$ finite and non-empty for all $x \in X$. Show that $\tilde{X}$ is compact Hausdorff if and only if X is compact Hausdorff.

9.9. Homotopies are evenly covered ⚙. Let $p\colon X \to Y$ be a covering map, and let $F\colon [0,1] \times [0,1] \to Y$ be a homotopy between two paths. For each $y \in Y$, let U_y be an evenly covered neighborhood of y. Show that there is $n > 0$ such that, subdividing $[0,1] \times [0,1]$ is squares of side length $1/n$, we obtained that the image under F of every such sub-square is contained in U_y for some $y \in Y$.

9.10. Complements of circles and lines ⊛. Let $\gamma := \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 = 1, z = 0\}$ and let $r := \{(x, y, z) \in \mathbb{R}^3 : x = y = 0\}$. Prove that the spaces $\mathbb{R}^3 \setminus \gamma$ and $\mathbb{R}^3 \setminus (\gamma \cup r)$ are not homeomorphic.

9. Solutions

Solution of 9.1: Let (x, y) be a point of S^1 and let $t \in \mathbb{R}$ be such that $x = \cos(2\pi t)$ and $y = \sin(2\pi t)$, i.e. $p(t) = (x, y)$. It is a well-known fact that $p^{-1}((x, y)) = \{t + n : n \in \mathbb{Z}\}$. Let $\varepsilon = 1/10$, we claim that the open set $U := p((t - \varepsilon, t + \varepsilon)) = \{(\cos(2\pi s), \sin(2\pi s)) : s \in (t - \varepsilon, t + \varepsilon)\}$ is an evenly covered open neighborhood of (x, y), which concludes the proof by arbitrariness of (x, y).

It is clear that $p^{-1}(U) = \bigcup_{n\in\mathbb{Z}} V_n$, where $V_n := (t + n - \varepsilon, t + n + \varepsilon)$. By the choice of ε, all the intervals V_n are disjoint. Fix $n \in \mathbb{Z}$, we want to show that $p|_{V_n}$ is a homeomorphism between V_n and U. Note that $p(s + n) = p(s)$ for all $s \in V_0$ and $n \in \mathbb{Z}$. Hence it is sufficient to prove that $p|_{V_0} : V_0 \to U$ is a homeomorphism.

Actually it is convenient to rather prove that $p|_{\overline{V_0}} : \overline{V_0} \to \overline{U}$ is a homeomorphism, which is a bit easier and from which follows that $p|_{V_0}$ is a homeomorphism with its image. It is clear that $p|_{\overline{V_0}}$ is a continuous bijection. Moreover we now prove that $p|_{\overline{V_0}}$ is closed, which concludes the proof. Consider a closed subset $C \subseteq \overline{V_0}$, then C is compact in $\mathbb{R}$ (since $\overline{V_0}$ is compact in $\mathbb{R}$). As a result, $p(C)$ is compact as well, thus closed since S^1 is Hausdorff.

Solution of 9.2: First note that the nth roots of unity, i.e. the solutions of $z^n = 1$, are $\xi_k := e^{2\pi ik/n} = \cos(2\pi k/n) + i \sin(2\pi k/n)$ for $k = 1, \dots, n$. Hence, given any $z = e^{i\theta} \in S^1$, we have that

$$p^{-1}(z) = \{e^{i\theta/n}\xi_1, e^{i\theta/n}\xi_2, \dots, e^{i\theta/n}\xi_n\}.$$

Now fix any $z_0 = e^{i\theta_0} \in S^1$, we claim that $U := \{e^{i\theta} : |\theta - \theta_0| < \pi/2\} \subseteq S^1$ is an evenly covered open neighborhood of z_0. Thanks to the observation above, $p^{-1}(U) = \bigcup_{k=1}^{n} V_k$, where

$$V_k := \{e^{i\theta/n}\xi_k : |\theta - \theta_0| < \pi/2\}.$$

Note that $V_k \cap V_h = \emptyset$ for all $k \neq h$. Indeed, assume by contradiction that $e^{i\theta/n}\xi_k = e^{i\theta'/n}\xi_h$ for some $\theta, \theta' \in (\theta_0 - \pi/2, \theta_0 + \pi/2)$ and $k \neq h$, then

$$\frac{\theta}{n} + \frac{2\pi k}{n} \equiv \frac{\theta'}{n} + \frac{2\pi h}{n} \pmod{2\pi}$$

$$\iff \theta - \theta' \equiv 2\pi(h - k) \pmod{2\pi n}.$$

However $|\theta - \theta'| < \pi < 2\pi|h - k| < 2\pi n$, which is in contradiction with the previous equality modulo $2\pi n$. Hence $\{V_k\}_{k=1,\dots,n}$ are pairwise disjoint. Moreover $p|_{V_k} : V_k \to U$ is easily a continuous bijection for every $k = 1, \dots, n$. Finally note that $(p|_{V_k})^{-1} : U \to V_k$ is given by

$$(p|_{V_k})^{-1}(e^{i\theta}) = e^{i\theta/n}\xi_k$$

for all $e^{i\theta} \in U$, i.e. for $|\theta - \theta_0| < \pi/2$. Hence $(p|_{V_k})^{-1}$ is continuous as well, which proves that $p|_{V_k} : V_k \to U$ is a homeomorphism for all $k = 1, \dots, n$ as we wanted.

Solution of 9.3: The map $p : \mathbb{C}^* \to \mathbb{C}^*$ is holomorphic with complex derivative $p'(z) = nz^{n-1}$. Now recall that the determinant of the Jacobian matrix of p (seen as a function from $\mathbb{R}^2 \setminus \{0,0\}$ to itself) in a point z is equal to $|p'(z)| = n|z|^{n-1}$, which is always different from zero in $\mathbb{C}^*$. Hence, we can apply the inverse function theorem to obtain that for all $z \in \mathbb{C}^*$ there exists an open neighborhood $V_z \subseteq \mathbb{C}^*$ of z such that $p|_{V_z} : V_z \to p(V_z) \subseteq \mathbb{C}^*$ is a homeomorphism.

Now consider any point $w \in \mathbb{C}^*$, then $p^{-1}(w) = \{z_1, \ldots, z_n\}$ consists of n points (if we write $w = re^{i\theta}$, then $p^{-1}(w) = \{r^{1/n}e^{i\theta/n}\xi_1, \ldots, r^{1/n}e^{i\theta/n}\xi_n\}$, where $\xi_1, \ldots, \xi_n$ are the nth roots of unity defined in Problem 9.2). Hence, p is a local homeomorphism with fibers of finite cardinatilty. Therefore p is a covering map since $\mathbb{C}^*$ is connected (see Lecture 18).

Remark. As we observed above, given any $w = re^{i\theta} \in \mathbb{C}^*$, we have that

$$p^{-1}(w) = \{r^{1/n}e^{i\theta/n}\xi_1, \ldots, r^{1/n}e^{i\theta/n}\xi_n\}.$$

Hence you can see that p winds only with respect to the variable θ. This makes an approach very similar to the one in Problem 9.2 working also in this case and providing a different proof from the one above. Indeed, it is possible to show in the same way as in the previous exercise that, given any point $w = r_0e^{i\theta_0} \in \mathbb{C}^*$, the set $U := \{re^{i\theta} : r > 0, |\theta - \theta_0| < \pi/2\} \subseteq \mathbb{C}^*$ is an evenly covered open neighborhood of w.

Solution of 9.4: First observe that a basis $\mathcal{B}$ of X is also a cover. Indeed the set X is open (for every topology on X), hence it can be written as union of elements in $\mathcal{B}$. Therefore, in order to prove both items, it is sufficient to prove that:

- If there is a cover of X consisting of evenly covered open sets, then p is a covering map.
- If p is a covering map, then there is a basis of X consisting of evenly covered open sets.

The first of the two statements is trivial. Indeed, if there is a cover $\mathcal{O}$ of X consisting of evenly covered open sets, then for all $x \in X$ there exists an evenly covered open neighborhood $U_x \in \mathcal{O}$ of x. However, this is exactly what is needed for p to be a covering map.

Hence let us prove the second statement. Consider the following family of open sets

$$\mathcal{B} := \{B \subseteq X : B \text{ is an evenly covered open set}\}.$$

We claim that $\mathcal{B}$ is a basis for the topology of X, which proves what we want. Consider any open subset O of X and pick a point $x \in O$. Since p is a covering map, there exists $B \in \mathcal{B}$ that contains x. Now observe that $B \cap O$ is an open neighborhood of x contained in O. Moreover $B \cap O$ is evenly covered, since it is a subset of an evenly covered set. Hence, $B \cap O \in \mathcal{B}$ and $x \in B \cap O \subseteq O$, which proves that $\mathcal{B}$ is a basis, as we wanted.

Finally, we prove that every covering map $p : \tilde{X} \to X$ is open. Let $U \subseteq \tilde{X}$ be an open set, we want to show that $p(U)$ is open in X. By the previous part of the exercise, there exists a cover $\mathcal{O}$ of X consisting of evenly covered open sets. Hence, given any $O \in \mathcal{O}$, we have that $p^{-1}(O) = \bigcup_{i \in I} V_i$, where V_i are disjoint open sets in $\tilde{X}$ such that $p|_{V_i} \colon V_i \to O$ is a homeomorphism. As a result, we obtain that

$$p(U) \cap O = p(U \cap (\cup_{i \in I} V_i)) = \bigcup_{i \in I} p(U \cap V_i)$$

is an open subset of X, since it is union of the open sets $p(U \cap V_i)$ for $i \in I$. Indeed, $U \cap V_i \subseteq \tilde{X}$ is open for all $i \in I$ and $p|_{V_i}$ maps it homeomorphically to $p(U \cap V_i)$, which is thus open as well. Therefore, we conclude that

$$p(U) = \bigcup_{O \in \mathcal{O}} (p(U) \cap O)$$

is open, as we wanted.

Solution of 9.5: Fix any point $x \in A$. Since $p \colon \tilde{X} \to X$ is a covering map, there exists an evenly covered open neighborhood U of x in X. Namely $p^{-1}(U) = \bigcup_{i \in I} V_i$, where V_i are disjoint open sets and $p|_{V_i} \colon V_i \to U$ is a homeomorphism for all $i \in I$.

We claim that $U^A := U \cap A$ is an evenly covered open neighborhood of x in A with respect to the map $p|_{\tilde{A}} \colon \tilde{A} \to A$. First observe that $(p|_{\tilde{A}})^{-1}(U^A) = p^{-1}(U^A) = \cup_{i \in I} V_i^A$, where $V_i^A := V_i \cap \tilde{A}$ are pairwise disjoint subsets of $\tilde{A}$. Moreover $p|_{V_i^A} \colon V_i^A \to p(V_i^A) = U^A$ maps homeomorphically V_i^A into U^A, since it is the restriction to V_i^A of the homeomorphism $p|_{V_i} \colon V_i \to U$. This concludes the proof by arbitrariness of $x \in A$.

Solution of 9.6: Let $z \in Z$, let $y_1, \ldots, y_n$ be the preimages of z with respect to the map q (by assumption they are in finite number), and let U be an evenly covered neighborhood of z (with respect to the cover $q \colon Y \to Z$). For all $i = 1, \ldots, n$, let U_i be an evenly covered open neighborhood of y_i (with respect to the cover $p \colon X \to Y$) and define $V_i := U_i \cap q^{-1}(U)$. Note that, for every $i = 1, \ldots, n$, we have that:

- V_i is an open neighborhood of y_i;
- $q|_{V_i}$ is a homeomorphism between V_i and $q(V_i)$;
- V_i is an evenly covered neighborhood for $p \colon X \to Y$.

Now, we can define $W \subseteq Z$ as

$$W := \bigcap_{i=1}^{n} q(V_i).$$

Since n is finite, W is an open subset of Z. Moreover, since W is contained in an evenly covered neighborhood of z (with respect to $q \colon Y \to Z$), we have that W is an evenly covered neighborhood of z as well. On the other hand, since $q^{-1}(W) \subseteq \bigcup_{i=1}^{n} V_i$, we have that $q^{-1}(W) = \bigcup_{i=1}^{n} W_i$, where W_i is contained in V_i for all $i = 1, \ldots, n$, thus W_i are disjoint evenly covered neighborhoods of y_i (with respect to p).

We claim that W is an evenly covered open neighborhood of z with respect to $q \circ p$. Note that $(q \circ p)^{-1}(V) = \bigcup_{i=1}^{n} p^{-1}(W_i)$, where $p^{-1}(W_i)$ are pairwise disjoint (since W_i are pairwise disjoint). Moreover, since every W_i is an evenly covered neighborhood for the covering map

p, we have that $p^{-1}(W_i) = \bigcup_{j\in J_i} T_i^j$, where $T_j^i \subseteq X$ are disjoint subsets and $p|_{T_i^j}: T_i^j \to W_i$ is a homeomorphism. As a result we have that $(q\circ p)|_{T_i^j} = q|_{W_i} \circ p|_{T_i^j}: T_i^j \to W$ is a homeomorphism, because it is the composition of two homeomorphism. Thus, W is an evenly covered neighborhood of $z \in Z$ with respect to $q \circ p$. Since $z \in Z$ was arbitrary, this concludes the proof.

Solution of 9.7: Fix any $x \in X$. We want to show that for all $y \in \tilde{X}$ there exists an open neighborhood U of y such that $(U \setminus \{y\}) \cap p^{-1}(x) = \emptyset$. We have to distinguish two cases:

- If $y \in p^{-1}(x)$, pick an evenly covered open neighborhood V of x. Then $p^{-1}(V) = \bigcup_{i\in I} U_i$, where $U_i \subseteq \tilde{X}$ are disjoint sets mapped homeomorphically to V. In particular, the sets U_i are open and for all $i \in I$ there is a unique $y_i \in p^{-1}(x) \cap U_i$. Hence we can define $U := U_i$, where $y \in U_i$, which is an open neighborhood of y such that $(U \setminus \{y\}) \cap p^{-1}(x) = \emptyset$.
- If $p(y) \neq x$, consider an open neighborhood V of $p(y)$ that does not contain x, which exists because X is Hausdorff (in fact it is sufficient to ask X to be T1). Up to restriction to a smaller neighborhood, we can assume that V is evenly covered. Then observe that $U := p^{-1}(V)$ is an open neighborhood of y that does not contain any element of $p^{-1}(x)$ (since it is union of sets mapped homeomorphically to V and $x \notin V$). In particular we have that $(U \setminus \{y\}) \cap p^{-1}(x) = \emptyset$.

Now assume that $\tilde{X}$ is compact and consider any point $x \in X$. In the previous part of the exercise we proved that $p^{-1}(x)$ is a discrete subset of $\tilde{X}$. However, any discrete subset of a compact topological space is finite, hence $p^{-1}(x)$ is finite for all $x \in X$. Note that the hypothesis on the connectivity of X is needed only to have that the degree is well-defined.

Thus we conclude by showing that any discrete subset of a compact topological space is finite. Assume by contradiction that there exists an infinite discrete subset A of a compact topological space Y. By definition of discrete subset, every $y \in Y$ admits an open neighborhood U_y such that $(U_y \setminus \{y\}) \cap A = \emptyset$. By compactness of Y, the open cover $\{U_y\}_{y\in Y}$ admits a finite subcover $\{U_{y_1}, \dots, U_{y_n}\}$. As a result, we obtain that

$$\emptyset = ((U_{y_1} \setminus \{y_1\}) \cup \ldots \cup (U_{y_n} \setminus \{y_n\})) \cap A \supseteq ((U_{y_1} \cup \ldots \cup U_{y_n}) \setminus \{y_1, \dots, y_n\}) \cap A,$$

which implies that $A \subseteq \{y_1, \dots, y_n\}$ is finite.

Solution of 9.8: First assume that $\tilde{X}$ is compact Hausdorff. Then $X = p(\tilde{X})$ is compact since it is the image of a compact set through a continuous function. Now consider any couple of distinct points $x, y \in X$. Let $p^{-1}(x) = \{\tilde{x}_1, \dots, \tilde{x}_d\}$ and $p^{-1}(y) = \{\tilde{y}_1, \dots, \tilde{y}_d\}$ be the preimages of x, y, where d is the (finite) degree of the cover. Then, since $\tilde{X}$ is Hausdorff, there exist pairwise disjoint open neighborhoods $\tilde{U}_1 \ni \tilde{x}_1, \dots, \tilde{U}_d \ni \tilde{x}_d$ and $\tilde{V}_1 \ni \tilde{y}_1, \dots, \tilde{V}_d \ni \tilde{y}_d$, namely $\tilde{U}_i \cap \tilde{V}_j = \emptyset$, $\tilde{U}_i \cap \tilde{U}_j = \emptyset$ and $\tilde{V}_i \cap \tilde{V}_j = \emptyset$ for all $1 \le i, j \le d$. Moreover, up to taking these neighborhoods $\tilde{U}_1, \dots, \tilde{U}_d, \tilde{V}_1, \dots \tilde{V}_d$ possibly smaller, we can assume that $p|_{\tilde{U}_i}: \tilde{U}_i \to U$ and $p|_{\tilde{V}_i}: \tilde{V}_i \to V$ are homeomorphisms for all $i = 1, \dots, d$, for some evenly covered open neighborhoods U, V of x, y respectively. We claim that U and

V are disjoint. Assume that there exists a point $z \in U \cap V$, then z has a preimage in each of the open subsets $U_1, \ldots, U_d, V_1, \ldots, V_d$, thus has at least $2d$ preimages (because these subsets are pairwise disjoint). This contradicts the fact that the degree of the cover is d and thus proves that X is Hausdorff, as desired.

Viceversa, let us assume that X is compact Hausdorff. Pick any couple of distinct points $\tilde{x}, \tilde{y} \in \tilde{X}$. Consider $x = p(\tilde{x})$ and $y = p(\tilde{y})$. If $x = y$, consider an evenly covered open neighborhood U_x of x, then $p^{-1}(U_x)$ is the disjoint union of sets homeomorphic to U_x. In particular, there exist $V_{\tilde{x}}, V_{\tilde{y}} \subseteq p^{-1}(U_x)$ disjoint open sets homeomorphic to U_x such that $\tilde{x} \in V_{\tilde{x}}$, $\tilde{y} \in V_{\tilde{y}}$. Now consider the case in which $x \neq y$, then there exist disjoint open neighborhoods U_x, U_y of x and y respectively. Up to intersecting them with evenly covered open neighborhoods of x and y, we can assume that U_x and U_y are evenly covered theirselves. Hence $p^{-1}(U_x)$ and $p^{-1}(U_y)$ are disjoint open neighborhoods of $\tilde{x}$ and $\tilde{y}$, respectively. This concludes the proof that $\tilde{X}$ is Hausdorff.

To prove that $\tilde{X}$ is compact, we will need the following basic lemma (cf. Lecture 9 and Lecture 10).

Lemma. *Let X be a compact Hausdorff space, then X is* regular*, i.e. for any $x \in X$ and any closed subset $C \subseteq X \setminus \{x\}$ there exist U, V disjoint open neighborhoods of x and C respectively.*

Proof. Since X is Hausdorff, for every $y \in C$ there exist U_y, V_y disjoint open neighborhoods of x and y, respectively. Moreover note that C is compact, because it is a closed subset of a compact Hausdorff space. Hence the open cover $\{V_y\}_{y \in C}$ of C admits a finite subcover $\{V_{y_1}, \ldots, V_{y_n}\}$. Hence we can define the open subsets $U := \bigcap_{k=1}^n U_{y_k}$ and $V := \bigcup_{k=1}^n V_{y_k}$. Observe that U and V are disjoint and they contain x and C respectively, which proves that X is regular. □

For every point $x \in X$, let us consider an evenly covered open neighborhood U_x of x. Then, by the lemma above applied to x and $X \setminus U_x$, we find an open neighborhood V_x of x such that $V_x \subseteq \overline{V_x} \subseteq U_x$. Note that V_x and $\overline{V_x}$ are evenly covered sets as well. Since X is compact, the open cover $\{V_x\}_{x \in X}$ admits a finite subcover $\{V_{x_1}, \ldots, V_{x_n}\}_{n \in \mathbb{N}}$. Now observe that $p^{-1}(\overline{V_{x_i}})$ is compact for all $i = 1, \ldots, n$, because it is a disjoint finite union of sets homeomorphic to $\overline{V_{x_i}}$, which is compact. Moreover, it holds that

$$\tilde{X} \supseteq \bigcup_{i=1}^n p^{-1}(\overline{V_{x_i}}) \supseteq \bigcup_{i=1}^n p^{-1}(V_{x_i}) = p^{-1}\left(\bigcup_{i=1}^n V_{x_i}\right) = p^{-1}(X) = \tilde{X}.$$

Hence all the containments are equalities and in particular $\tilde{X}$ is the finite union of the compact sets $p^{-1}(\overline{V_{x_i}})$ for $i = 1, \ldots, n$, thus it is compact as well.

Solution of 9.9: Observe that $\mathcal{O} = \{F^{-1}(U_y)\}_{y \in Y}$ is an open cover of $[0,1] \times [0,1]$, which is a compact metric space. Thus the cover $\mathcal{O}$ admits a Lebesgue number, i.e. there exists $\varepsilon > 0$ such that for each $x \in [0,1] \times [0,1]$ there is $O \subseteq \mathcal{O}$ with $B_\varepsilon(x) \subseteq O$.

Now pick $n > 0$ such that $2/n < \varepsilon$. We claim that subdividing the square in sub-squares of side length $1/n$ provides the result. Let Q be such a sub-square and let x be a point in Q. By definition of Lebesgue number, there exists $O \in \mathcal{O}$ such that $B_\varepsilon(x) \subseteq O$. By the choice of n, this imiplies that $Q \subseteq B_\varepsilon(x) \subseteq O$. Since $O = F^{-1}(U_y)$ for some $y \in Y$, we have that $F(Q) \subseteq U_y$, which is the desired result.

Solution of 9.10: