10. Computing the fundamental group - Part I

Chef's table

This week we start computing fundamental group, mainly using Van Kampen's Theorem (but possibly in combination with other tools). The first two problems are sort of basic warm-up exercises to get a feeling for the subject. Problems 10.3 - 10.4 - 10.5 are almost identical, and they all build on the same trick (think in terms of the planar models!); you can write down the solution to just one of them, but make sure you give some thought about all. From there (so building on these results), using Van Kampen you can compute the fundamental group of the torus (which we knew anyway, but through a different method), of the Klein bottle and of higher-genus surfaces. Problem 10.8 is the most important in this series, and learning this trick will trivialise half of the problems on this subject (see Problem 10.9 for a first, striking, application); writing down all homotopies of 10.8 explicitly might be a bit tedious, so just make sure to have a clear picture (and keep in mind this result for the future).

10.1. Topological manifold minus a point. Let X be a connected topological manifold, of dimension $n \geq 3$. Prove that for every $x \in X$ one has $\pi_1(X) \cong \pi_1(X \setminus \{x\})$.

10.2. Plane without the circle. Show that there is no homeomorphism $f\colon \mathbb{R}^2 \setminus S^1 \to \mathbb{R}^2 \setminus S^1$ such that $f(0,0) = (2,0)$.

10.3. Torus minus a point. Compute the fundamental group of $T^2 \setminus \{p\}$, where T^2 stands for the standard torus and p is any point in T^2.

10.4. Klein bottle minus a point. Compute the fundamental group of $K^2 \setminus \{p\}$, where K^2 stands for the Klein bottle and p is any point in K^2.

10.5. Surface minus a point. For any $g \geq 2$, let Σ_g denote a genus g closed orientable surface. Compute the fundamental group of $\Sigma_g \setminus \{p\}$.

Note: It may be useful to use the following definition of the genus g surface Σ_g. Let P_{4g} be a (say, regular) $4g$-sided polygon, whose sides are enumerated from 0 to $4g-1$ in clockwise order. Then Σ_g is the surface obtain from P_{4g} identifying the side S parametrized in clockwise direction with the side $S+2$ parametrized in counterclockwise direction, for all $S = 0, 4, 8, \ldots, 4(g-1)$ and for all $S = 1, 5, 9, \ldots, 4(g-1)+1$.

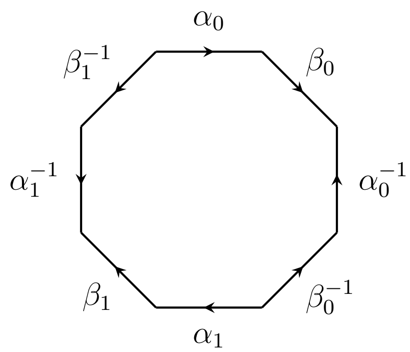

Figure 1: Example of the construction of Σ_g by identification of the sides of a $4g$-sided polygon in the case $g = 2$.

10.6. Application of Van Kampen's Theorem. Relying on the result of Problems 10.3 and 10.4, employ Van Kampen's Theorem to compute $\pi_1(T^2)$ and $\pi_1(K^2)$.

10.7. Fundamental group of a surface. Relying on the result of Problem 10.5, employ Van Kampen's Theorem to compute $\pi_1(\Sigma_g)$.

10.8. Bouquet of circles. Prove that the following three topological spaces are homotopy equivalent:

- $X_1 :=$ the wedge sum of k circles;
- $X_2 := \mathbb{R}^2$ minus k points;
- $X_3 :=$ the circle union with k of its radii.

Hence compute their fundamental group.

10.9. Euclidean space minus a finite number of points. Compute the fundamental group of $\mathbb{R}^n$ minus k points, as one varies $n \geq 2$ and $k \geq 0$. In particular, determine all pairs (n, k) so that such space is simply connected.

10.10. Edges of the cube. Compute the fundamental group of the space X that is the union of the edges of the cube $C = [0, 1]^3$. Moreover compute the fundamental group of $X \setminus \{x\}$ as one varies $x \in X$.

10. Solutions

Solution of 10.1: Fix a point $x \in X$ and consider an open neighborhood U of x that is homeomorphic to $\mathbb{R}^n$. Moreover define the open set $V := X \setminus \{x\}$. Then note that:

- $X = U \cap V$;
- U is simply connected, since it is homeomorphic to $\mathbb{R}^n$;
- $U \cup V = U \setminus \{x\}$ is simply connected, since it is homeomorphic to $\mathbb{R}^n$ minus a point, which deformation retracts onto S^{n-1} (see Problem 10.9 for the explicit deformation retraction) and thus it is simply connected for $n \geq 3$ (see Lecture 20).

As a result, by Van Kampen's Theorem we have that $\pi_1(X) = \pi_1(U) * \pi_1(V) = \pi_1(V) = \pi_1(X \setminus \{x\})$, as desired.

Solution of 10.2: Let us consider any homeomorphism $f\colon \mathbb{R}^2 \setminus S^1 \to \mathbb{R}^2 \setminus S^1$. Note that $\mathbb{R}^2 \setminus S^1$ has two connected components. Let us denote by A the unbounded connected component and B the bounded one (which corresponds with the open disc in $\mathbb{R}^2$). Since f is a homeomorphism, then the two connected components are either kept or switched by f. If $f(0,0) = (2,0)$, then f must switch A and B, since $(0,0) \in B$ and $(2,0) \in A$. In particular $f|_B\colon B \to A$ is a homeomorphism. However this is impossible because the fundamental group of B is trivial, since B is contractible, and the fundamental group of A is $\mathbb{Z}$, since A deformation retracts onto a circle.

Solution of 10.3: Let us represent the torus T^2 as a square with opposite sides identified, that is $Q/\sim$, where $Q := [-1,1] \times [-1,1]$ and $(x,-1) \sim (x,1)$ for $x \in [-1,1]$, $(-1,y) \sim (1,y)$ for $y \in [-1,1]$ (see Figure 2).

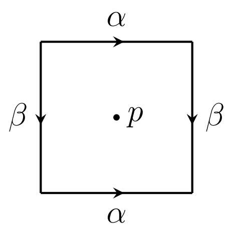

Figure 2: Torus as quotient of the square.

Moreover, we can assume that $p = (0,0)$ is the center of the square. Observe that $Q \setminus \{p\}$ deformation retracts on the boundary of the square ∂Q, via the homotopy

$$H((x,y),t) = (1-t)(x,y) + tb(x,y),$$

where $b(x,y) \in \partial Q$ is the unique intersection of ∂Q with the ray starting from p and going through (x,y). As a result, the fundamental group of $(Q/\sim) \setminus \{p\}$ is the same as the fundamental group of $\partial Q/\sim$. However, observe that $\partial Q/\sim$ is homeomorphic to the wedge sum of two circles, hence we obtain that

$$\pi_1(T^2 \setminus \{p\}) = \pi_1(\partial Q/\sim) = \pi_1(S^1 \vee S^1) = \pi_1(S^1) * \pi_1(S^1) = \mathbb{Z} * \mathbb{Z},$$

by Van Kampen's Theorem.

Solution of 10.4: Similarly to what we did in the proof of the previous exercise (in fact the whole argument will be very similar), let us represent the Klein bottle K^2 as a quotient of the square, that is $Q/\sim$, where $Q := [-1,1] \times [-1,1]$ and $(x,-1) \sim (x,1)$ for $x \in [-1,1]$, $(-1,y) \sim (1,-y)$ for $y \in [-1,1]$ (see Figure 3). Note that, differently from the torus, here the vertical sides are identified with opposite orientation.

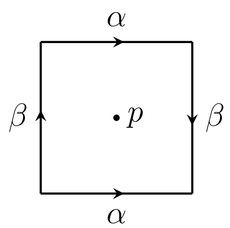

Figure 3: Klein bottle as quotient of the square.

We can assume that $p = (0,0)$ is the center of the square. Then, exactly the same homotopy as in Problem 10.1 shows that $(Q/\sim) \setminus \{p\}$ deformation retracts onto $\partial Q/\sim$, which is homeomorphic to the wedge sum of two circles (as before, even if the equivalent relation is different). Hence, by Van Kampen's Theorem, we obtain that

$$\pi_1(K^2 \setminus \{p\}) = \pi_1(\partial Q/\sim) = \pi_1(S^1 \vee S^1) = \pi_1(S^1) * \pi_1(S^1) = \mathbb{Z} * \mathbb{Z}.$$

Solution of 10.5: The solution retraces again the procedure presented in Problem 10.3, substituting the square with the $4g$-sided polygon. Assume that $p = (0,0)$ is the center of the $4g$-sided regular polygon P_{4g} and let us denote by $\sim$ the equivalent relation that identifies the sides of P_{4g} as in the statement, in such a way that $\Sigma_g = P_{4g}/\sim$. Then, $P_{4g} \setminus \{p\}$ deformation retracts on the boundary ∂P_{4g} of the polygon, via the homotopy

$$H((x,y),t) = (1-t)(x,y) + tb(x,y),$$

where $b(x,y) \in \partial P_{4g}$ is the unique intersection of ∂P_{4g} with the ray starting from p and going through (x,y). Therefore, the fundamental group of $\Sigma_g \setminus \{p\}$ is the same as the fundamental group of $\partial P_{4g}/\sim$. However, it is not difficult to check that $\partial P_{4g}/\sim$ is homeomorphic to the wedge sum of $2g$ copies of S^1, thus its fundamental group is the free product of $2g$ copies of $\mathbb{Z}$.

Solution of 10.6: As in Problem 10.3, we represent T^2 as $Q/\sim$, where $\sim$ is the equivalent relation defined in the aforementioned exercise. Consider the covering of T^2 consisting of the open sets $A = (Q/\sim)\setminus\{(0,0)\} \subseteq Q/\sim$ and $B = \mathrm{int}(Q) = (-1,1)\times(-1,1) \subseteq Q/\sim$. By Problem 10.3, the fundamental group of A is $\mathbb{Z} * \mathbb{Z}$, whose generators are $a = [\alpha], b = [\beta]$, where $\alpha, \beta\colon [0,1] \to A$ are the curves $\alpha(t) := (-1+2t, -1) = (-1+2t, 1)$ and $\beta(t) := (-1, 1-2t) = (1, 1-2t)$ (see Figure 2). On the other hand, B has trivial fundamental group. Moreover note that $A \cap B = ((-1,1)\times(-1,1)) \setminus \{(0,0)\}$ is homeomorphic to $S^1 \times (0,1)$, hence it has fundamental group $\mathbb{Z}$ and a generator for $\pi_1(A\cap B)$ is a curve homotopic to $\alpha * \beta * \alpha^{-1} * \beta^{-1}$ (i.e. a curve winding clockwise once around the origin).

As a result, by Van Kampen's Theorem, we have that

$$\pi_1(T^2) = {}^{(\pi_1(A) * \pi_1(B))}\!/_{N} = {}^{\pi_1(A)}\!/_{N} = {}^{(\mathbb{Z} * \mathbb{Z})}\!/_{N},$$

where N is the normal subgroup of $\mathbb{Z} * \mathbb{Z}$ generated by $aba^{-1}b^{-1}$. It remains to show that $(\mathbb{Z} * \mathbb{Z})/N$ is isomorphic to $\mathbb{Z}\times\mathbb{Z}$. Consider the homomorphism $F\colon \mathbb{Z} * \mathbb{Z} \to \mathbb{Z}\times\mathbb{Z}$, defined on the generators a, b as $F(a) = (1,0)$ and $F(b) = (0,1)$. Note that $F(aba^{-1}b^{-1}) = F(a) + F(b) - F(a) - F(b) = 0$, hence $N \subseteq \ker(F)$. Therefore the map F descends to the quotient as a homomorphism $\tilde{F}\colon (\mathbb{Z}\times\mathbb{Z})/N \to \mathbb{Z}\times\mathbb{Z}$. We want to show that $\tilde{F}$ is an isomorphism. The surjectivity follows easily from the definition, so let us prove the injectivity. Consider any $w \in \mathbb{Z}\times\mathbb{Z}$ such that $F(w) = 0$. Then w is a word with as many a as a^{-1} and as many b as b^{-1}. Note that, if $w = xbay$ for some words x and y, then $w(y^{-1}a^{-1}b^{-1}aby) = xaby$ and

$$y^{-1}a^{-1}b^{-1}aby = (y^{-1}b^{-1}a^{-1})(aba^{-1}b^{-1})(aby) = (aby)^{-1}(aba^{-1}b^{-1})(aby) \in N.$$

Therefore $[w] = [xbay] = [xaby]$ in $(\mathbb{Z} * \mathbb{Z})/N$. Equivalently one can check that, if $w = xb^{-1}a^{-1}y$ for some words x and y, then $[w] = [xb^{-1}a^{-1}y] = [xa^{-1}b^{-1}y]$ in $(\mathbb{Z} * \mathbb{Z})/N$. As a result, $[w] = [v]$, where v is a word with all a, a^{-1} to the left and all b, b^{-1} to the right, in the same number as in w. However since the number of a is the same as the number of a^{-1} (in w and so in v), and the same holds for b, b^{-1}, then v is the empty word (because a cancels with a^{-1} and b cancels with b^{-1}). Thus $[w] = 0 \in (\mathbb{Z} * \mathbb{Z})/N$, which prove that $\ker(F) = \{0\}$, hence F is an isomorphism.

Let us now consider the case of the Klein bottle $K^2 = Q/\sim$. Similarly as before (see also Figure 3), we define $A = (Q/\sim)\setminus\{(0,0)\}$, $B = \mathrm{int}(Q)$ and the generators of $\pi_1(A) = \mathbb{Z} * \mathbb{Z}$ as $a = [\alpha], b = [\beta]$, where $\alpha, \beta\colon [0,1] \to A$ are the curves $\alpha(t) := (-1+2t, -1) = (-1+2t, 1)$ and $\beta(t) := (1, 1-2t) = (-1, -1+2t)$ (note that now the quotient on the boundary of Q is different). Then, a generator for the fundamental group of $A \cap B$ is $\alpha * \beta * \alpha^{-1} * \beta$. Then, by Van Kampen's Theorem, we have that

$$\pi_1(K^2) = {}^{(\pi_1(A) * \pi_1(B))}\!/_{N} = {}^{\pi_1(A)}\!/_{N} = {}^{(\mathbb{Z} * \mathbb{Z})}\!/_{N},$$

where N is the normal subgroup of $\mathbb{Z} * \mathbb{Z}$ generated by $aba^{-1}b$. Unfortunately, in this case, we do not have a better expression for this group.

Solution of 10.7: The solution follows the same lines of argument as in Problem 10.6, in light of Problem 10.5. Let us represent Σ_g as $P_{4g}/\sim$. Moreover consider the open sets $A := (P_{4g}/\sim) \setminus \{(0,0)\} \subseteq \Sigma_g$ and $B := \mathrm{int}(P_{4g}) \subseteq \Sigma_g$, which together cover Σ_g. By

Problem 10.5, the fundamental group of A is the free product of $2g$ copies of $\mathbb{Z}$, while $\pi_1(B)$ is trivial. Moreover observe that the fundamental group of $A \cap B$ is $\mathbb{Z}$ and a choice of generator is a curve homotopic to

$$\gamma := (\alpha_0 * \beta_0 * \alpha_0^{-1} * \beta_0^{-1}) * (\alpha_1 * \beta_1 * \alpha_1^{-1} * \beta_1^{-1}) * \ldots * (\alpha_{g-1} * \beta_{g-1} * \alpha_{g-1}^{-1} * \beta_{g-1}^{-1}),$$

where α_k, β_k are parametrizations of the sides $4k$ and $4k+1$ of P_{4g} for $k = 0, \ldots, g-1$. Note that γ is a parametrization of $\partial P_{4g}/\sim$. See Figure 1 for a representation of the case $g = 2$.

As a result, by Van Kampen's Theorem, we obtain that

$$\pi_1(\Sigma_g) = {(\pi_1(A) * \pi_1(B))}/{N} = {\pi_1(A)}/{N} = {(\mathbb{Z} * \ldots * \mathbb{Z})}/{N},$$

where N is the normal subgroup of $\mathbb{Z} * \ldots * \mathbb{Z}$ generated by

$$[a_0, b_0][a_1, b_1] \ldots [a_{g-1}, b_{g-1}],$$

where $a_k := [\alpha_k] \in \pi_1(A)$, $b_k := [\beta_k] \in \pi_1(A)$ for all $k = 0, \ldots, g-1$ and $[a_k, b_k] := a_k b_k a_k^{-1} b_k^{-1} \in \pi_1(A)$.

Solution of 10.8: We will show that X_2 deformation retracts to both X_1 and X_3, which proves that the three spaces are homotopy equivalent. Their fundamental group is then the fundamental group of the wedge sum of k circles, namely the free product of k copies of $\mathbb{Z}$.

Without loss of generality we can assume that $X_2 = \mathbb{R}^2 \setminus \{p_1, \ldots, p_k\}$, where $p_1, \ldots, p_k$ are distinct points on S^1. Let $\gamma_1, \ldots, \gamma_k$ be simple closed curves going through the origin such that p_i is contained in the bounded component of $\mathbb{R}^2 \setminus \gamma_i$ for all $i = 1, \ldots, k$ and $\gamma_i \cap \gamma_j = \{(0,0)\}$ for all $i \neq j$ (see Figure 4 on the left). Note that $\gamma_1 \cup \ldots \cup \gamma_k$ is homeomorphic to X_1. Moreover it is not difficult to check that X_2 deformation retracts to X_1, sending the bounded component of $(\mathbb{R}^2 \setminus \{p_1, \ldots, p_k\}) \setminus \gamma_i$ (which is homeomorphic to a disk minus a point, the point p_i) to γ_i and the unbounded component to all X_1.

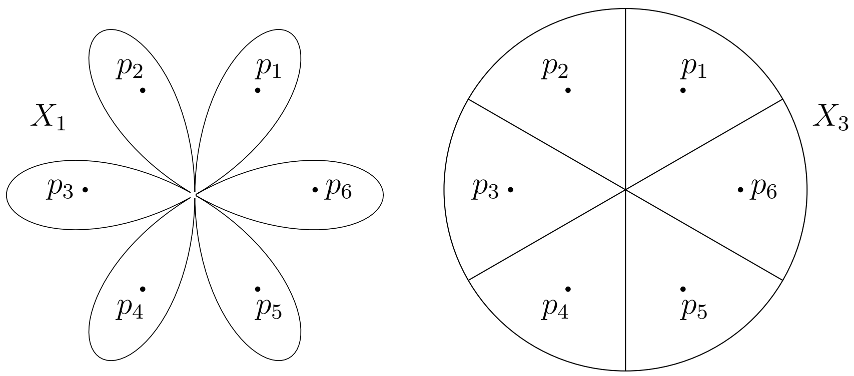

Figure 4: Construction of X_1 and X_3 with respect to the points $p_1, \ldots, p_k$.

Now let X_3 be the circle of radius 2 in $\mathbb{R}^2$ with radii $l_1, \ldots, l_k$ such that $p_1, \ldots, p_k$ are contained in different connected components of $\mathbb{R}^2 \setminus X_3$ (i.e. each radius stays between two of the points, see Figure 4 on the right). As before, it is not difficult to check that X_2 deformation retracts to X_3, concluding the proof.

Solution of 10.9: For all $n \geq 2, k \geq 1$, we want to compute the fundamental group $G_{n,k}$ of $\mathbb{R}^n$ minus k points. In particular we will prove by induction on k that $G_{n,k}$ is homomorphic to the free product of k copies of $\mathbb{Z}$ if $n = 2$ and it is trivial for $n \geq 3$.

Let us first consider the base cases $k = 0$ and $k = 1$. The Euclidean space $\mathbb{R}^n$ is contractible for all $n \geq 1$, hence its fundamental group $G_{n,0}$ its trivial. Let us now compute the fundamental group of $\mathbb{R}^n$ minus a point p for $n \geq 2$. Without loss of generality we can assume that p is the origin. Then note that $\mathbb{R}^n \setminus \{p\}$ deformation retracts on S^{n-1} via the homotopy $H\colon (\mathbb{R}^n \setminus \{p\}) \times [0,1] \to S^{n-1}$ given by $H(x,t) = (1-t)x + tx/\|x\|$. As a result, we obtain that

$$G_{n,1} = \pi_1(\mathbb{R}^n \setminus \{p\}) = \pi_1(S^{n-1}) = \begin{cases} \mathbb{Z} & \text{for } n = 2 \\ \{0\} & \text{for } n \geq 3. \end{cases}$$

Let us now show the inductive step. Consider $k \geq 2$ distinct points $p_1, \dots, p_k$ in $\mathbb{R}^n$. Let H be a hyperplane in $\mathbb{R}^n$ that does not contain any of the points $p_1, \dots, p_k$ and such that p_1 and p_2 are contained in different connected components of $\mathbb{R}^n \setminus H$. Since the points are in finite number, this hyperplane exists. Without loss of generality we can assume that $H = \{x_1 = 0\}$, where we consider standard coordinates $(x_1, \dots, x_n) \in \mathbb{R}^n$, and that $p_1 \in \{x_1 > 0\}$, $p_2 \in \{x_1 < 0\}$.

Now note that there is $\varepsilon > 0$ such that $H_\varepsilon := \{(x_1, \dots, x_n) \in \mathbb{R}^n : |x_1| < \varepsilon\}$ is again disjoint from $\{p_1, \dots, p_k\}$. Then consider the open subsets of $\mathbb{R}^n \setminus \{p_1, \dots, p_k\}$ given by

$$\begin{aligned} A &:= \{(x_1, \dots, x_n) \in \mathbb{R}^n : x_1 > -\varepsilon\} \setminus \{p_1, \dots, p_k\} \\ B &:= \{(x_1, \dots, x_n) \in \mathbb{R}^n : x_1 < \varepsilon\} \setminus \{p_1, \dots, p_k\}. \end{aligned}$$

Observe that $A \cup B = \mathbb{R}^n \setminus \{p_1, \dots, p_k\}$ and $A \cap B = H_\varepsilon$ is simply connected. Let k_A, k_B be the number of points $p_1, \dots, p_k$ contained in A and B, respectively. Since none of the points is contained in $A \cap B$, it holds $k_A + k_B = k$. Moreover $1 \leq k_A < k$ and $1 \leq k_B < k$, because $p_1 \in A$ and $p_2 \in B$. Note that the sets A and B are homeomorphic to $\mathbb{R}^n$ minus k_A and k_B points, respectively. Hence, $\pi_1(A) = G_{n,k_A}$ and $\pi_1(B) = G_{n,k_B}$. We can thus apply Van Kampen's Theorem, using the inductive hypothesis, and obtain that

$$G_{2,k} = \pi_1(\mathbb{R}^2 \setminus \{p_1, \dots, p_k\}) = \pi_1(A) * \pi_1(B) = (\underbrace{\mathbb{Z} * \ldots * \mathbb{Z}}_{k_A \text{ copies}}) * (\underbrace{\mathbb{Z} * \ldots * \mathbb{Z}}_{k_B \text{ copies}}) = \underbrace{\mathbb{Z} * \ldots * \mathbb{Z}}_{k \text{ copies}}$$

and that, for $n \geq 3$, instead we have

$$G_{n,k} = \pi_1(\mathbb{R}^n \setminus \{p_1, \dots, p_k\}) = \pi_1(A) * \pi_1(B) = \{0\} * \{0\} = \{0\},$$

which is exactly what we wanted.

Solution of 10.10: