11. Computing fundamental group - Part II

Chef's table

In this (second to last) problem set, we keep training on themes related to the fundamental group of a topological space. The first two problems are two (very instructive!) exercises on the correspondence between Algebra (structure of free groups) on one side and Topology (structure of wedges) on the other side. It is always helpful, both here and at a more advanced level, to reinterpret facts on either side of the mirror. Problems from 11.3 to 11.8 are six exercises where you ultimately need to combine suitable retractions (to reduce those spaces to simpler ones) and, possibly, Van Kampen's Theorem applied to correctly chosen sets (the art is always to pick the correct sets, which makes the whole difference between a straightforward conclusion and an insane mess). Problem 11.9 provides a topological proof of the Fundamental Theorem of Algebra (can you list 5 different proofs of this result?). Lastly, Problem 11.10 is… quite a challenge! Try it at your own risk.

11.1. Finite order words ✍. Determine all and only elements of finite order in the free group F_n with $n \geq 1$ generators.

11.2. Non-isomorphic free groups ✍.

(i) Prove, through a topological argument, that the wedge of m circles is not homeomorphic to the wedge of n circles if $m \neq n$.

(ii) Prove that the group F_m is not isomorphic to F_n if $m \neq n$, hence deduce that the wedge of m circles is not homotopic to the wedge of n circles if $m \neq n$.

11.3. Plane and circle ✍. Compute the fundamental group of the following subspaces of $\mathbb{R}^3$:

(i) $X_1 := \{(0, y, z) \in \mathbb{R}^3\} \cup \{(x, y, 0) \in \mathbb{R}^3 \,:\, x^2 + y^2 = 1,\ x \geq 0\}$, i.e., the union of a plane with a half-circle with end points on the plane;

(ii) $X_2 := \{(0, y, z) \in \mathbb{R}^3\} \cup \{(x, y, 0) \in \mathbb{R}^3 \,:\, x^2 + y^2 = 1\}$, i.e., the union of a plane with a circle that intersects the plane transversely in two points.

11.4. Difference of linear spaces ✍. Compute the fundamental group of $V \setminus W$, where V is a linear space over $\mathbb{R}$ of dimension $n \geq 2$ and $W \subset V$ is a linear subspace of dimension k with $0 \leq k \leq n - 2$.

11.5. Intersecting planes ⚙. Let X be the topological subspace of $\mathbb{R}^3$ given by the union of three distinct planes (not necessarily passing through the origin). Compute the fundamental group in any configuration such that X is connected.

11.6. Union of sphere and planes ⚙. Compute $\pi_1(X)$, where X is the subspace of $\mathbb{R}^3$ obtained as the union of the unit sphere and the three coordinate planes, namely

$$X := S^2 \cup \{x = 0\} \cup \{y = 0\} \cup \{z = 0\}.$$

11.7. Complement of two linked circles. Compute the fundamental group of the complement of two linked circles in the unit sphere $S^3 \subseteq \mathbb{R}^4$. Same question for $\mathbb{R}^3$ in lieu of S^3.

11.8. Subspace of the two-dimensional complex space. Compute the fundamental group of the subspace X of $\mathbb{C}^2$ defined as

$$X := \{(z, w) \in \mathbb{C}^2 : |z| = |w| = 1\} \cup \{(1, w) \in \mathbb{C}^2\}.$$

11.9. Fundamental Theorem of Algebra. Let $p(z) = z^n + a_{n-1}z^{n-1} + \ldots + a_0$ with $n > 0$ be a complex polynomial with positive degree. We want to show that p admits at least one root.

(i) Let us first assume that $\|a_0\| + \ldots + \|a_{n-1}\| < 1$.

  (a) Let $f\colon S^1 \to \mathbb{C}^*$ be the map defined as $f(z) := z^n$. Show that f is not null-homotopic.

  (b) Show that there is a homotopy H between f and the restriction of p on S^1, such that H has values in $\mathbb{C}^*$.

  (c) Assume, by contradiction, that p does not admit any root. Show that this implies that p is null-homotopic in $\mathbb{C}^*$ and obtain a contradiction.

(ii) Reduce the general case to the case when $\|a_0\| + \ldots + \|a_{n-1}\| < 1$ and prove the Fundamental Theorem of Algebra.

11.10. Complex lines through the origin. Compute the fundamental group of $\mathbb{C}^2 \setminus (\ell_1 \cup \ell_2)$ where ℓ_1, ℓ_2 are two distinct complex lines through the origin.

11. Solutions

Solution of 11.1: We want to show that the only finite order element in F_n is the trivial one, i.e., the empty word.

Let $a_1, \dots, a_n$ be the alphabet (i.e., the generators) of the free group F_n and consider a finite order word w, namely w^k is the empty word for some $k \geq 1$. If $k = 1$, then w is the empty word, thus let us assume $k > 1$ and w non-trivial. Moreover assume that $w = a_{i_1} \dots a_{i_l}$ is written in reduced form, for some $l \geq 1$ and $i_1, \dots i_l \in \{1, \dots, n\}$. We prove by induction on the length l of w that w is trivial. If $l = 0$ and $l = 1$, this is obvious. Thus assume $l > 1$. If w^k is the trivial word, then

$$(a_{i_1} \dots a_{i_l})(a_{i_1} \dots a_{i_l}) \dots (a_{i_1} \dots a_{i_l}) = \emptyset.$$

Since w is irreducible, we must have that $a_{i_l} a_{i_1} = \emptyset$ and that

$$a_{i_1}(a_{i_2} \dots a_{i_{l-1}})(a_{i_2} \dots a_{i_{l-1}}) \dots (a_{i_2} \dots a_{i_{l-1}}) a_{i_l} = a_{i_1}(a_{i_2} \dots a_{i_{l-1}})^k a_{i_l} \emptyset.$$

In particular we have $(a_{i_2} \dots a_{i_{l-1}})^k = \emptyset$. Hence, by inductive hypothesis (the length of $a_{i_2} \dots a_{i_{l-1}}$ is $l - 2$) we have that $a_{i_2} \dots a_{i_{l-1}} = \emptyset$. Therefore we obtain $w = a_{i_1} a_{i_l} = \emptyset$, as desired.

Solution of 11.2:

(i) Let W_m be the wedge sum of m circles and let $p_m \in W_m$ be the point such that all the circles go through p_m. Observe that p_m is the only point that disconnects W_m and $W_m \setminus \{p_m\}$ consists of m connected components. As a result, W_m is not homeomorphic to W_n for every $m \neq n$, since the property of having a unique point disconnecting and the number of connected components in which it disconnects are topological invariants.

(ii) Without loss of generality assume that $m < n$. Suppose by contradiction that there is an isomorphism $\phi\colon F_m \to F_n$. Moreover consider a surjective homomorphism $\psi\colon F_n \to \mathbb{Z}^n$, which easily exists (for example sending generators $g_1, \dots, g_n$ of F_n in $(1, 0 \dots, 0), (0, 1, 0, \dots, 0), \dots, (0, \dots, 0, 1)$). Then let $h_1, \dots, h_m$ be generators of F_m. Note that both ϕ and ψ are surjective homomorphisms, thus $\psi(\phi(h_1)), \dots, \psi(\phi(h_m))$ generates $\mathbb{Z}^n$, but this is not possible since $m < n$.

As a result, we obtain that the wedge of m circles is not homotopically equivalent to the wedge of n circles for $m \neq n$, because otherwise they would have the same fundamental group.

Solution of 11.3: First, let us observe that the plane $P := \{(0, y, z) \in \mathbb{R}^3\}$ deformation retracts onto the segment $s := \{(0, y, 0) \in \mathbb{R}^3 : |y| \leq 1\}$. Indeed P deformation retracts to the line $\ell := \{(0, y, 0) \in \mathbb{R}^3\}$ via the homotopy $H\colon P \times [0, 1] \to P$ such that

$$H((0, y, z), t) = (0, y, (1 - 2t)z)$$

and the line ℓ deformation retracts onto the segment s via the homotopy $G\colon \ell \times [0,1] \to \ell$ given by

$$G((0,y,0),t) = \begin{cases} (0,y,0) & \text{if } |y| \leq 1 \\ (0,(1-t)y + ty/|y|, 0) & \text{if } |y| > 1. \end{cases}$$

(i) Thanks to the observation above, the space X_1 deformation retracts onto the union of the segment s and the half-circle $\hat{\gamma} := \{(x,y,0) \in \mathbb{R}^3 : x^2 + y^2 = 1,\ x \geq 0\}$. Indeed, the deformation retraction defined above and acting on $P = \{(0,y,z) \in \mathbb{R}^3\}$ keeps the end points $(0,-1,0)$ and $(0,1,0)$ of the half-circle fixed. Now note that $s \cup \hat{\gamma}$ is homeomorphic to a circle, hence $\pi_1(X_1) = \mathbb{Z}$.

(ii) As in (i), it holds that X_2 deformation retracts to the union of the segment s and the circle $\gamma := \{(x,y,0) \in \mathbb{R}^3 : x^2 + y^2 = 1\}$. Note that s is a diameter of the circle γ, hence the fundamental group of $s \cup \gamma$ is $\mathbb{Z} * \mathbb{Z}$ (see Problem 10.8). As a result we obtain that $\pi_1(X_2) = \mathbb{Z} * \mathbb{Z}$.

Solution of 11.4: Without loss of generality we can assume that $V = \mathbb{R}^n$ and $W = \{(x_1, \ldots, x_n) \in \mathbb{R}^n : x_{k+1} = \ldots = x_n = 0\}$. Then $V \setminus W$ deformation retracts to $X := \{(x_1, \ldots, x_n) \in \mathbb{R}^n : x_1 = \ldots = x_k = 0\} \setminus \{(0, \ldots, 0)\}$ via the homotopy $H\colon V \setminus W \times [0,1] \to V \setminus W$ given by

$$H((x_1, \ldots, x_n), t) := (tx_1, \ldots, tx_k, x_{k+1}, \ldots, x_n).$$

Observe that X is homeomorphic to $\mathbb{R}^{n-k}$ minus a point p, therefore

$$\pi_1(V \setminus W) = \pi_1(X) = \pi_1(\mathbb{R}^{n-k} \setminus \{p\}) = \begin{cases} \mathbb{Z} & \text{if } n-k = 2 \\ \{0\} & \text{if } n-k \geq 2. \end{cases}$$

Remark. We required $k \leq n-2$, because if $k = n-1$ then $V \setminus W$ is disconnected.

Solution of 11.5: If the three planes are parallel, then X is disconnected. Hence at most two of the planes can be parallel. Let us distinguish three cases:

- *Two of the planes are parallel and the third is transverse (i.e., not parallel) to them.* Let P_1, P_2 be the parallel planes and P_3 the other one. Note that P_1, P_2 easily deformation retract to the lines $P_1 \cap P_3$ and $P_2 \cap P_3$, respectively. Therefore $X = P_1 \cup P_2 \cup P_3$ deformation retracts to P_3, which is contractible. Therefore $\pi_1(X) = \{0\}$.
- *The planes are all transverse one to the other and they all pass through a point.* Let us denote by p the common intersection point. Then X deformation retracts to p, simply via the homotopy $H\colon X \times [0,1] \to X$ given by $H(x,t) = (1-t)x + tp$. Therefore we obtain again that $\pi_1(X) = \{0\}$.

- *The planes are all transverse one to the other without a common intersection point.* Let us denote by P_1, P_2, P_3 the planes and by $l_{12} = P_1 \cap P_2$, $l_{23} = P_2 \cap P_3$, $l_{13} = P_1 \cap P_3$ the intersection lines. Then l_{12}, l_{23} and l_{13} are parallel (i.e., have the same direction), because if two of the lines intersect then also the third one must go through the same point, falling in the previous case. Let Q be a plane orthogonal to l_{12}, l_{23} and l_{13}. Then X deformation retracts to $X \cap Q$ via orthogonal projection. Now note that $X \cap Q$ is the union of three lines pairwise intersecting at distinct points (i.e., $l_{12} \cap Q$, $l_{23} \cap Q$ and $l_{13} \cap Q$), hence it deformation retracts to a triangle, which has fundamental group $\mathbb{Z}$. As a result, in this case we obtain $\pi_1(X) = \mathbb{Z}$.

Solution of 11.6: First observe that X is path-connected, hence it makes sense to talk about the fundamental group without specifying a base point. Let $Y := X \cap \overline{B^3}$ be the space obtained intersecting X with the closed unit ball $\overline{B^3} = \{(x, y, z) \in \mathbb{R}^3 : \|(x, y, z)\| \leq 1\}$ of $\mathbb{R}^3$. Note that Y is homotopic equivalent to X, since $H\colon X \times [0, 1] \to X$ defined as

$$H((x, y, z), t) := \begin{cases} (x, y, z) & \text{if } (x, y, z) \in Y \\ \dfrac{(x, y, z)}{(1 - t) + t\|(x, y, z)\|} & \text{if } (x, y, z) \in X \setminus Y \end{cases}$$

is a deformation retraction of X to Y. Thus, we only need to compute the fundamental group of Y.

Consider the subset $A_1 := \{(x, y, z) \in Y : x, y, z > -1/3\}$ of Y. It is easy to see that A_1 can be retracted to $\{(x, y, z) \in Y : x, y, z \geq 0\}$, which is equal to the sector of the sphere contained in the first quadrant $\{x, y, z \geq 0\}$ union with the three quarters of a disk $\{(0, y, z) : y, z \geq 0\} \cap \overline{B^3}$, $\{(x, 0, z) : x, z \geq 0\} \cap \overline{B^3}$ and $\{(x, y, 0) : x, y \geq 0\} \cap \overline{B^3}$. In particular $\{(x, y, z) \in Y : x, y, z \geq 0\}$ is homeomorphic to a sphere, hence A_1 is simply connected.

Now consider $A_2 := \{(x, y, z) \in Y : x, y > -1/3\}$, for which it holds

$$A_2 = \{(x, y, z) \in Y : x, y, z > -1/3\} \cup \{(x, y, z) \in Y : x, y > -1/3,\ z < 1/3\}.$$

Note that the first set of the union is A_1, while the second set is just the symmetric of A_1 with respect to the reflection $z \mapsto -z$. In particular they are both simply connected. Moreover

$$\begin{aligned}\{(x, y, z) \in Y : x, y, z > -1/3\} &\cap \{(x, y, z) \in Y : x, y > -1/3,\ z < 1/3\} \\ &= \{(x, y, z) \in Y : x, y > -1/3,\ -1/3 < z < 1/3\}\end{aligned}$$

deformation retracts on $\{(x, y, 0) \in Y : x, y \geq 0\}$, which is a quarter of a disk, thus simply connected. As a result, by Van Kampen's Theorem, we obtain that A_2 is simply connected.

Then we define $A_3 := \{(x, y, z) \in Y : x > -1/3\}$. Again we can write A_3 as

$$A_3 = \{(x, y, z) \in Y : x, y > -1/3\} \cup \{(x, y, z) \in Y : x > -1/3,\ y < 1/3\}.$$

The first set of the union is A_2 and the second set is the symmetric of A_2 with respect to the reflection $y \mapsto -y$, thus they are both simply connected. Similarly as before, we can also infer that the intersection is simply connected, so it is A_3 by Van Kampen's Theorem.

Finally we repeat the same procedure once again writing

$$Y = \{(x, y, z) \in Y \,:\, x > -1/3\} \cup \{(x, y, z) \in Y \,:\, x < 1/3\}.$$

The first set is A_3 and again we can go through the same argument and obtain that Y is simply connected, which concludes the exercise.

Solution of 11.7: Let $\gamma_1, \gamma_2 \subseteq S^3$ be two linked circles in S^3 and let $p \in \gamma_2$ be a point contained in one of these two circles. Recalling that $S^3 \setminus \{p\}$ is homeomorphic to $\mathbb{R}^3$, it is easy to see that $S^3 \setminus \{\gamma_1 \cup \gamma_2\} = (S^3 \setminus \{p\}) \setminus \{\gamma_1 \cup (\gamma_2 \setminus \{p\})\}$ is homeomorphic to the complement of a circle and a line passing inside the circle in $\mathbb{R}^3$, i.e., $\mathbb{R}^3 \setminus \{\gamma \cup r\}$ in the notation of Problem 9.10. The fundamental group of $\mathbb{R}^3 \setminus \{\gamma \cup r\}$ is $\mathbb{Z}$ (see Exercise class of May 4th), hence it is so also the fundamental group of the complement of two linked circles in S^3.

Now observe that the complement of two linked circles in $\mathbb{R}^3$ is homeomorphic to $(S^3 \setminus \{\gamma_1 \cup \gamma_2\}) \setminus \{x\}$, where x is a point in $S^3 \setminus \{\gamma_1 \cup \gamma_2\}$. Note that $S^3 \setminus (\gamma_1 \cup \gamma_2)$ is an open subset of the 3-manifold S^3, so we can apply Problem 10.1 and obtain that the fundamental group of the complement of two linked circles in $\mathbb{R}^3$ is equal to

$$\pi_1((S^3 \setminus \{\gamma_1 \cup \gamma_2\}) \setminus \{x\}) = \pi_1(S^3 \setminus \{\gamma_1 \cup \gamma_2\}) = \mathbb{Z}.$$

Solution of 11.8: Let us first observe that X deformation retracts onto the space

$$Y := \{(z, w) \in \mathbb{C}^2 \,:\, |z| = |w| = 1\} \cup \{(1, w) \in \mathbb{C}^2 \,:\, |w| \leq 1\}$$

via the homotopy $H \colon X \times [0, 1] \to X$ given by

$$H((z, w), t) = \begin{cases} (z, w) & \text{if } |w| \leq 1 \\ (z, (1-t)w + tw/|w|) & \text{if } |w| > 1. \end{cases}$$

Note that Y is a torus union with a disk in its interior. Hence consider the open subsets Y_1, Y_2 of Y defined as

$$Y_1 := \{(z, w) \in Y \,:\, |w| > 1 - 1/2\}, \quad Y_2 := \{(z, w) \in Y \,:\, |z - 1| < 1/2\}.$$

Note that $Y = Y_1 \cup Y_2$. Moreover Y_1 deformation retracts to the torus $\{(z, w) \in \mathbb{C}^2 : |z| = |w| = 1\}$, Y_2 deformation retracts to the disk $\{(1, w) \in \mathbb{C}^2 \,:\, |w| \leq 1\}$ and $Y_1 \cap Y_2$ deformation retracts to the circle $\{(1, w) \in \mathbb{C}^2 \,:\, |w| = 1\}$. As a result, if we see $\pi_1(Y_1 \cap Y_2) = \mathbb{Z}$ as the subgroup of $\pi_1(Y_1) = \mathbb{Z} \times \mathbb{Z}$ generating the second factor, by Van Kampen's Theorem, we obtain that

$$\pi_1(X) = \pi_1(Y) = \pi_1(Y_1 \cup Y_2) = {}^{\pi_1(Y_1) * \pi_1(Y_2)}\big/_{\pi_1(Y_1 \cap Y_2)} = {}^{\mathbb{Z} \times \mathbb{Z}}\big/_{\mathbb{Z}} = \mathbb{Z}.$$

Solution of 11.9:

(i) (a) It is easy to see that the map $G\colon \mathbb{C}^* \times [0,1] \to S^1$ defined as

$$G\colon (z,t) \mapsto \frac{z}{(1-t)+t\|z\|}$$

is a deformation retraction of $\mathbb{C}^*$ on S^1. In particular we have that $\pi_1(\mathbb{C}^*) = \pi_1(S^1) = \mathbb{Z}$.

Now note that

$$f(e^{2\pi it}) = e^{2\pi nit} = \cos(2\pi nt) + i\sin(2\pi nt),$$

for all $t \in [0,1]$. Hence, the class of f in $\pi_1(S^1)$ is not trivial (see Lecture 18) and thus the homotopy class of f in $\pi_1(\mathbb{C}^*)$ is not trivial as well, because of the isomorphism $\pi_1(\mathbb{C}^*) = \pi^1(S^1)$ above.

(b) Consider the map $H\colon S^1 \times [0,1] \to \mathbb{C}$ defined as

$$H(z,t) := z^n + t(a_{n-1}z^{n-1} + \ldots + a_0),$$

which is clearly a homotopy between f and p. We need to show that H takes values in $\mathbb{C}^*$. For every $z \in S^1$ and $t \in [0,1]$, we have

$$\begin{aligned}\|H(z,t)\| &= \|z^n + t(a_{n-1}z^{n-1} + \ldots + a_0)\| \geq \|z^n\| - \|t(a_{n-1}z^{n-1} + \ldots + a_0)\| \\ &\geq 1 - t(\|a_{n-1}z^{n-1}\| + \ldots + \|a_0\|) \geq 1 - t(\|a_{n-1}\| + \ldots + \|a_0\|) > 0,\end{aligned}$$

which proves that $H(z,t) \in \mathbb{C}^*$, as desired.

(c) Assume by contradiction that p does not admit any root, then $p(\mathbb{C})$ is contained in $\mathbb{C}^*$. In particular $F\colon S^1 \times [0,1] \to \mathbb{C}^*$ given by

$$F(z,t) := p((1-t)z)$$

provides a homotopy between $p|_{S^1}\colon S^1 \to \mathbb{C}^*$ and the constant $p(0)$. Hence p is null-homotopic as a map in $\mathbb{C}^*$, which contradicts the fact that p is homotopic to f.

(ii) Let $c > 0$ be any real constant. Note that z_0 is a root of p if and only if z_0/c is a root of the polynomial

$$q(z) := z^n + \frac{a_{n-1}}{c}z^{n-1} + \frac{a_{n-2}}{c^2}z^{n-1} + \ldots + \frac{a_0}{c^n}.$$

Now note that it is possible to choose c sufficiently large such that

$$\left\|\frac{a_{n-1}}{c}\right\| + \left\|\frac{a_{n-2}}{c^2}\right\| + \ldots + \left\|\frac{a_0}{c^n}\right\| = \frac{\|a_{n-1}\|}{c} + \frac{\|a_{n-2}\|}{c^2} + \ldots + \frac{\|a_0\|}{c^n} < 1.$$

Hence, for this choice of c, q admits a root by (i) and thus p admits a root as well.

Solution of 11.10: