12. Coverings, reloaded

Chef's table

This final problem set is centered around the interplay of coverings and fundamental group. It is a good review of many/most of the techniques we have seen along the semester, and you will have two weeks to solve it.

Some of the exercises are centered around the construction of simply-connected covering spaces (i.e., of the universal cover of a given topological space): Problem 12.5 revisits one key technical point of the general construction; Problem 12.3 (but also, from different perspectives, Problem 12.4, and the challenge Problem 12.10) shows that in concrete cases one can *forget about the abstract proof and perform these constructions with bare hands*; while Problem 12.6 shows that if the base space X is locally too wild then the universal cover just does not exist (hence one cannot really weaken the assumptions in our theorem).

(The second part of) Problem 12.7 is a famous (and somewhat surprising!) result in Topology: *on the Earth there are always two antipodal locations having the same temperature and pressure.* This theorem is proven by lifting the right curves. Problem 12.8 is also a famous result, which follows straight from an application of 12.7 by simply choosing the right function f. Lastly, Problem 12.9 is what I would call an 'exercise of style', which concerns calculating, with two different techniques (either using Van Kampen or using pure covering arguments), the fundamental group of a certain space: I find it very instructive in that it sheds some light on the strengths and weaknesses of these two methods.

12.1. Compact universal cover and functions to the circle. Let X be a (locally path-connected) topological space having compact universal cover. Prove that any continuous function $f\colon X \to S^1$ is homotopic to a constant.

12.2. Compact universal cover and other coverings. Prove that a topological space having compact universal cover admits finitely many covering spaces up to isomorphism.

12.3. Coverings of the torus. Let T^2 be the standard torus. For every subgroup H of $\pi_1(T^2)$, find a covering map $q\colon X \to T^2$ such that $\operatorname{Im}(q_*) = H$.

Hint: Note that, since $\pi_1(T^2) = \mathbb{Z} \times \mathbb{Z}$ is an abelian group, every covering space of T^2 is normal.

12.4. Different coverings of the torus. Determine two covering maps $p\colon X \to T^2$ and $p'\colon X' \to T^2$ of the standard torus T^2 such that:

- $p\colon X \to T^2$ and $p'\colon X' \to T^2$ have the same (finite) number of sheets;

- there do not exist homeomorphisms $\phi\colon X \to X'$ and $\psi\colon T^2 \to T^2$ with $p' \circ \phi = \psi \circ p$.

12.5. Construction of the universal cover - technical details. In this problem we wish to revisit one technical point in the construction of the universal cover of a topological space. Let then X be a topological space (assumed to be path-connected, locally path-connected and semi-locally simply-connected), and fix $x_0 \in X$. In class (Lecture 24) we have defined a simply-connected topological space $\tilde{X}$ by declaring

$$\tilde{X} := \{[\gamma] \,:\, \gamma\colon I \to X \text{ satisfies } \gamma(0) = x_0\},$$

where $[\gamma]$ denotes the equivalence class under homotopies preserving both endpoints; we also defined the surjective map $p\colon \tilde{X} \to X$ by $p([\gamma]) := \gamma(1)$. Check that p is continuous, and then that p is in fact a covering map.

12.6. When the universal cover does not exist. Let X be the subspace of $\mathbb{R}^2$ consisting of the four sides of the square $[0,1] \times [0,1]$ together with the segments of the vertical lines $x = 1/2, 1/3, 1/4, \ldots$ inside the square. Show that for every covering space $p\colon \tilde{X} \to X$ there is some neighborhood of the left edge $\{0\} \times [0,1]$ of X that lifts homeomorphically to $\tilde{X}$. Deduce that X has no simply-connected covering space.

12.7. Functions on antipodal points. Prove the following versions of the Borsuk-Ulam theorem.

(i) Prove that any continuous map $f\colon S^2 \to \mathbb{R}$ must attain the same value at a pair of antipodal points.

(ii) Prove that any continuous map $f\colon S^2 \to \mathbb{R}^2$ must attain the same value at a pair of antipodal points.

12.8. Covering of the sphere with closed sets. Let $A_1, \ldots, A_k$ be closed subsets of S^2 whose union is S^2 itself. Prove that if $k \leq 3$ then there exists $i \in \{1, \ldots, k\}$ such that A_i contains a pair of antipodal points. How about $k \geq 4$?

12.9. Projective space minus one point. Compute, in two different ways (directly via Van Kampen's Theorem, and alternatively using covering arguments) the fundamental group of $\mathbb{P}^2(\mathbb{R})$ minus one point.

12.10. Explicit construction of universal covers. Construct a simply-connected covering space of the space $X \subseteq \mathbb{R}^3$ that is the union of a sphere and a diameter. Do the same when X is the union of a sphere and a circle intersecting it in two points.

12. Solutions

Solution of 12.1: Since the universal cover of X is compact, it has finite degree by Problem 9.8, thus $\pi_1(X)$ is finite (see Proposition 2 in Lecture 23). As a result, any element $h \in \pi_1(X)$ has finite order. Therefore $f_*(h) \in \pi_1(S^1) = \mathbb{Z}$ has finite order as well (because f_* is a homomorphism), which implies that it is zero since there are no non-trivial finite order elements in $\mathbb{Z}$. Thus we proved that $f_*(\pi_1(X)) = \{0\} \subseteq \pi_1(S^1)$.

Now consider the universal cover $p\colon \mathbb{R} \to S^1$ of S^1. Since $f_*(\pi_1(X)) = \{0\} = p_*(\pi_1(\mathbb{R}))$, the assumptions of the lifting criterion are satisfied and thus there exists a lift $\tilde{f}\colon X \to \mathbb{R}$ of f. Now, since X is compact, $\tilde{f}$ is easily homotopic to a constant, an explicit homotopy $H\colon X \times [0,1] \to \mathbb{R}$ being given by $H(x,t) = (1-t)\tilde{f}(x) + t$. Hence, $G\colon X \times [0,1] \to S^1$ defined as $G(x,t) = p(H(x,t))$ is a homotopy between $f = p \circ \tilde{f}$ and the constant function.

Solution of 12.2: Let $x_0 \in X$ be any point, then $\pi_1(X, x_0)$ is a finite group because the universal cover of X is compact. Now consider any subgroup $H < \pi_1(X, x_0)$ and two covers $p_1\colon \tilde{X}_1 \to X$ and $p_2\colon \tilde{X}_2 \to X$ such that $(p_1)_*(\pi_1(\tilde{X}_1, x_1)) = (p_2)_*(\pi_1(\tilde{X}_2, x_2)) = H$ for points $x_1 \in \tilde{X}_1$ e $x_2 \in \tilde{X}_2$. Then, by Proposition 1.37 in [AT], $p_1\colon \tilde{X}_1 \to X$ and $p_2\colon \tilde{X}_2 \to X$ are isomorphic, via an isomorphism $f\colon \tilde{X}_1 \to \tilde{X}_2$ that maps x_1 to x_2. As a result, the number of coverings spaces of X (up to isomorphism) is at most equal to the number of subgroups of $\pi_1(X, x_0)$, which is finite since $\pi_1(X, x_0)$ is finite.

Solution of 12.3: Let $p\colon \mathbb{R}^2 \to T^2$ be the universal cover of $T^2 = S^1 \times S^1$ given by

$$p(t,s) := (\cos(2\pi t), \sin(2\pi t), \cos(2\pi s), \sin(2\pi s)).$$

We know that $\pi_1(T^2)$ is isomorphic to the group $G(\mathbb{R}^2, p)$ of deck transformations of the covering, via $\varphi\colon \pi_1(T^2) = \mathbb{Z}^2 \to G(\mathbb{R}^2, p)$ defined as $\varphi(n,m)(t,s) = (t+n, s+m)$ for all $(n,m) \in \mathbb{Z}^2$, $(t,s) \in \mathbb{R}^2$. Observe that φ is such that for every loop $\gamma\colon [0,1] \to T^2$ it holds $\varphi([\gamma])(\tilde{\gamma}(0)) = \tilde{\gamma}(1)$, where $\tilde{\gamma}\colon [0,1] \to \mathbb{R}^2$ is a lift of γ.

The group $\varphi(H)$ is a normal subgroup of $G(\mathbb{R}^2, p)$ isomorphic to H, thus we can consider the quotient $X := \mathbb{R}^2/\varphi(H)$ of $\mathbb{R}^2$ with respect to the automorphism group $\varphi(H)$. Moreover define the map $q\colon X \to T^2$ as $q([x]) = p(x)$ for all $x \in \mathbb{R}^2$, where $[x]$ denotes its equivalent class in X. This map is well-defined because, if $[x] = [y]$ for some $x, y \in \mathbb{R}^2$, then $p(x) = p(y)$. Moreover note that $q \circ r = p$, where $r\colon \mathbb{R}^2 \to X$ is the quotient map.

It is easy to check that $q\colon X \to T^2$ is a covering map, thus it remains only to prove that $q_*(\pi_1(X)) = H$. Take any element $[\gamma] \neq 0 \in \pi_1(T^2)$, where $\gamma\colon [0,1] \to T^2$ is a loop in T^2, and consider a lift $\tilde{\gamma}\colon [0,1] \to \mathbb{R}^2$ of γ. Note that $r \circ \tilde{\gamma}\colon [0,1] \to X$ is a (non-trivial) loop if and only if $\varphi([\gamma]) \in \varphi(H)$, i.e., if and only if $[\gamma] \in H$. Moreover $q_*([r \circ \tilde{\gamma}]) = [q \circ r \circ \tilde{\gamma}] = [p \circ \tilde{\gamma}] = [\gamma] \in H$, which concludes the proof.

Solution of 12.4: Let us identify the torus T^2 with $\mathbb{R}^2/\mathbb{Z}^2$ and consider the covering spaces $p\colon T^2 \to T^2$ and $p'\colon T^2 \to T^2$ given by $p([(t,s)]) := [(2t, 2s)]$ and $p'([(t,s)]) := [(t, 4s)]$ for every $[(t,s)] \in T^2 = \mathbb{R}^2/\mathbb{Z}^2$. Observe that p and p' are both covering spaces of degree 4 over T^2. Moreover note that $p_*(\pi_1(T^2)) = (2\mathbb{Z}) \times (2\mathbb{Z}) < \mathbb{Z} \times \mathbb{Z} = \pi_1(T^2)$ and $p_*(\pi_1(T^2)) = \mathbb{Z} \times (4\mathbb{Z}) < \mathbb{Z} \times \mathbb{Z} = \pi^1(T^2)$.

Now assume by contradiction that there exist homeomorphisms $\phi\colon X \to X'$ and $\psi\colon T^2 \to T^2$ such that $p' \circ \phi = \psi \circ p$, then in particular

$$(2\mathbb{Z}) \times (2\mathbb{Z}) = p'_*(\mathbb{Z} \times \mathbb{Z}) = p'_* \circ \phi_*(\pi_1(T^2)) = \psi_* \circ p_*(\pi_1(T^2)) = \psi_*(\mathbb{Z} \times (4\mathbb{Z})),$$

i.e., there exists an isomorphism $F = \psi_*\colon \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z} \times \mathbb{Z}$ such that $F(\mathbb{Z} \times (4\mathbb{Z})) = (2\mathbb{Z}) \times (2\mathbb{Z})$. However, this is patently not possible for example because the quotients $(\mathbb{Z} \times \mathbb{Z})/(\mathbb{Z} \times (4\mathbb{Z})) = \mathbb{Z}/4\mathbb{Z}$ and $(\mathbb{Z} \times \mathbb{Z})/((2\mathbb{Z}) \times (2\mathbb{Z})) = (\mathbb{Z}/2\mathbb{Z}) \times (\mathbb{Z}/2\mathbb{Z})$ are not isomorphic.

Solution of 12.5: First let us recall the topology we have defined on $\tilde{X}$ in Lecture 24. Consider $\mathcal{U} := \{U \subseteq X$ path-connected open set s.t $\pi_1(U) \to \pi_1(X)$ is trivial$\}$, which is a basis for the topology of X as observed in class. Now, given $\gamma\colon I \to X$ with $\gamma(0) = x_0$ and $U \in \mathcal{U}$, we define

$$U_{[\gamma]} := \{[\gamma * \eta] \,:\, \eta \text{ is a path in } U \text{ with } \eta(0) = \gamma(1)\}$$

(note that $U_{[\gamma]} = \emptyset$ if $\gamma(1) \notin U$). Moreover denote by

$$\mathcal{O} := \{U_{[\gamma]} \,:\, \gamma\colon I \to X,\ \gamma(0) = x_0,\ U \in \mathcal{U}\}$$

the set of the sets $U_{[\gamma]}$ for all possible choices of γ and U as above. Let us check that $\mathcal{O}$ is the basis for a topology of $\tilde{X}$.

- Consider any $\gamma\colon I \to X$ with $\gamma(0) = x_0$. Since $\mathcal{U}$ is a covering of X, there is $U \in \mathcal{U}$ such that $\gamma(1) \in U$. Then we have that $[\gamma] \in U_{[\gamma]}$, which proves that $\mathcal{O}$ covers $\tilde{X}$.
- Consider two sets $U^1_{[\gamma_1]}, U^2_{[\gamma_2]} \in \mathcal{O}$ and assume that $U^1_{[\gamma_1]} \cap U^2_{[\gamma_2]}$ is non-empty, so there exists $[\gamma] \in U^1_{[\gamma_1]} \cap U^2_{[\gamma_2]}$, i.e., $[\gamma] = [\gamma_1 * \eta_1] = [\gamma_2 * \eta_2]$ with η_i path in U^i with $\eta_i(0) = \gamma_i(1)$, for $i = 1, 2$. Since we consider equivalence classes under homotopies preserving both endpoints, we have that $\gamma(1) = (\gamma_1 * \eta_1)(1) = (\gamma_2 * \eta_2)(1)$. However, note that $(\gamma_1 * \eta_1)(1) \in U^1$ and $(\gamma_2 * \eta_2)(1) \in U^2$, thus $\gamma(1) \in U^1 \cap U^2$. Now let $U \in \mathcal{U}$ be such that $\gamma(1) \in U \subseteq U^1 \cap U^2$, which exists because $\mathcal{U}$ is a basis. We claim that $[\gamma] \in U_{[\gamma]} \subseteq U^1_{[\gamma_1]} \cap U^2_{[\gamma_2]}$. Let η be any path in U with $\eta(0) = \gamma(1)$, in such a way that $[\gamma * \eta]$ is a generic element of $U_{[\gamma]}$. Then, for $i = 1, 2$, we have $[\gamma * \eta] = [(\gamma_i * \eta_i) * \eta] = [\gamma_i * (\eta_i * \eta)]$ and $\eta_i * \eta$ is a path in U^i with $(\eta_i * \eta)(0) = \gamma_i(1)$, thus $[\gamma * \eta] = [\gamma_i * (\eta_i * \eta)] \in U^i_{[\gamma_i]}$. This proves that $U_{[\gamma]} \subseteq U^1_{[\gamma_1]} \cap U^2_{[\gamma_2]}$, as we wanted.

We can now prove that $p\colon \tilde{X} \to X$ is continuous. Consider any $U \in \mathcal{U}$, then it is sufficient to check that $p^{-1}(U)$ is open in $\tilde{X}$. First note that $p^{-1}(U) = \{[\gamma] \in \tilde{X} \,:\, \gamma(1) \in U\}$, hence $U_{[\gamma]} \in \mathcal{O}$ is non-empty for all $[\gamma] \in p^{-1}(U)$, in particular it contains $[\gamma]$. We claim that

$$p^{-1}(U) = \bigcup_{[\gamma] \in p^{-1}(U)} U_{[\gamma]}, \tag{1}$$

from which follows that $p^{-1}(U)$ is open. Obviously the left-hand side is contained in the right-hand side, because for every $[\gamma] \in p^{-1}(U)$ it holds $[\gamma] \in U_{[\gamma]}$. Hence, let us prove the other inclusion. Given $[\gamma] \in p^{-1}(U)$, consider any $[\gamma * \eta] \in U_{[\gamma]}$, then $p([\gamma * \eta]) = (\gamma * \eta)(1) \in U$, which proves that $U_{[\gamma]} \subseteq p^{-1}(U)$. This shows that the right-hand side in (1) is contained in the left-hand side and thus concludes the proof of the continuity of p.

We can also show that p is open, since $p(U_{[\gamma]}) = U$ for every $U_{[\gamma]} \in \mathcal{O}$. Indeed, any $[\gamma * \eta] \in U_{[\gamma]}$ has end point $(\gamma * \eta)(1)$ in U, thus $p(U_{[\gamma]}) \subseteq U$. Moreover, for every point $x \in U$ there exists a curve η in U such that $\eta(0) = \gamma(1)$ and $\eta(1) = x$. Hence $[\gamma * \eta] \in U_{[\gamma]}$ and $p([\gamma * \eta]) = (\gamma * \eta)(1) = x$, which proves also that $p(U_{[\gamma]}) \supseteq U$.

It remains only to show that p is a covering map. In particular we will prove that U is an evenly covered open subset for all $U \in \mathcal{U}$. Given any $U \in \mathcal{U}$, observe that:

- $p\colon U_{[\gamma]} \to U$ is a homeomorphism. Indeed p is open and $p(U_{[\gamma]}) = U$, as observed above. Moreover we can show that $p\colon U_{[\gamma]} \to U$ is injective. Let η_1, η_2 be curves in U such that $\eta_1(0) = \eta_2(0) = \gamma(1)$ and $\eta_1(1) = \eta_2(1)$, i.e., $[\gamma * \eta_1], [\gamma * \eta_2] \in U_{[\gamma]}$ and $p([\gamma * \eta_1]) = p([\gamma * \eta_2])$. Since $\pi_1(U) \to \pi_1(X)$ is trivial and $\eta_1 * \eta_2^{-1}$ is a closed curve in U, we have that $[\eta_1 * \eta_2^{-1}]$ is the identity of $\pi_1(X)$, thus $[\eta_1] = [\eta_2]$ and $[\gamma * \eta_1] = [\gamma * \eta_2]$.
- For all $[\gamma_1], [\gamma_2] \in p^{-1}(U)$ we have that either $U_{[\gamma_1]} = U_{[\gamma_2]}$ or $U_{[\gamma_1]} \cap U_{[\gamma_2]} = \emptyset$. Indeed, if there exists $[\gamma_1 * \eta_1] = [\gamma_2 * \eta_2] \in U_{[\gamma_1]} \cap U_{[\gamma_2]}$, for some curves η_1, η_2 with $\eta_1(0) = \gamma_1(1)$, $\eta_2(0) = \gamma_2(1)$ and $\eta_1(1) = \eta_2(1)$, then we have that $[\gamma_2] = [\gamma_1 * (\eta_1 * \eta_2^{-1})]$. Thus, for any $[\gamma_2 * \eta] \in U_{[\gamma_2]}$, it holds $[\gamma_2 * \eta] = [\gamma_1 * (\eta_1 * \eta_2^{-1} * \eta)] \in U_{[\gamma_1]}$, which proves that $U_{[\gamma_2]} \subseteq U_{[\gamma_1]}$. The other inclusion is completely analogous.

These two observations together with (1) imply that U is an evenly covered set, as desired.

Solution of 12.6: Let us consider the curve $\gamma\colon [0,1] \to X$ given by $\gamma(t) := (0, t)$. Then take a lift $\tilde{\gamma}\colon [0,1] \to \tilde{X}$ of γ, which exists by the lifting criterion. Then, for every $t \in [0,1]$, let $U_t \subseteq X$ be an evenly covered open neighborhood of $\gamma(t)$ and let $V_t \subseteq \tilde{X}$ be such that $p\colon V_t \to U_t$ is a homeomorphism and $\tilde{\gamma}(t) \in V_t$. Hence define the open subsets $U := \bigcup_{t\in[0,1]} U_t$ and $V := \bigcup_{t\in[0,1]} V_t$ and observe that $p|_V\colon V \to U$ is a homeomorphism and U is an open neighborhood of $\{0\} \times [0,1]$.

Note that, since U is a neighborhood of $\{0\} \times [0,1]$, there exists k such that $([0, 1/k) \times [0,1]) \cap X \subseteq U$. Hence consider the open sets $A := (p|_V)^{-1}([0, 1/k) \times [0,1]) \subseteq V$ and $B := \tilde{X} \setminus (p|_V)^{-1}([0, 1/(k+2)] \times [0,1])$. Note that A and B are both open sets since $p|_V$ is a homeomorphism and $[0, 1/k) \times [0,1]$, $[0, 1/(k+2)] \times [0,1]$ are respectively open and close in U. Now note that $A \cap B = (p|_V)^{-1}((1/(k+2), 1/k) \times [0,1])$ is simply-connected and that $\tilde{X} = A \cup B$. Therefore, by Van Kampen's Theorem we obtain that $\pi_1(\tilde{X}) = \pi_1(A) * \pi_1(B)$. However $\pi_1(\tilde{X})$ is trivial and $\pi_1(A)$ is not, thus we have a contradiction.

Solution of 12.7:

(i) Consider the function $g\colon S^2 \to \mathbb{R}$ defined as $g(x) := f(x) - f(-x)$. The statement is equivalent to require that g attains value 0. Note that the image $g(S^2)$ of g is connected and symmetric with respect to the origin, because $g(x) = -g(-x)$; therefore it contains zero, as we wanted.

(ii) Assume by contradiction that $f(x) \neq f(-x)$ for all $x \in S^2$, then the function $g\colon S^2 \to S^1$ given by $g(x) := (f(x)-f(-x))/\|f(x)-f(-x)\|$ is well-defined and continuous. Now consider the curve $\gamma\colon [0,1] \to S^2$ given by $\gamma(s) := (\cos(2\pi s), \sin(2\pi s), 0)$, which parametrizes an equatorial curve in the sphere, and consider the function $h\colon [0,1] \to S^1$ obtained composing g with γ, i.e., $h := g \circ \gamma$. Thanks to the lifting criterion (since $[0,1]$ is simply connected), there is a lift $\tilde{h}\colon [0,1] \to \mathbb{R}$ of h with respect to the universal cover $p\colon \mathbb{R} \to S^1$ of S^1 defined as $p(t) := (\cos(2\pi t), \sin(2\pi t))$.

Observe that $g(x) = -g(-x)$, therefore

$$\begin{aligned} h(s+1/2) &= g((\cos(2\pi s+\pi), \sin(2\pi s+\pi))) = g(-(\cos(2\pi s), \sin(2\pi s))) \\ &= -g((\cos(2\pi s), \sin(2\pi s))) = -h(s), \end{aligned}$$

for all $s \in [0,1/2]$. This implies that $\tilde{h}(s+1/2) = \tilde{h}(s) + q(s)/2$, for some odd integer number $q(s)$, depending on $s \in [0,1/2]$. However, note that $\tilde{h}$ is continuous, thus $q(s)$ must be equal to a constant odd integer number q, since it takes values in a discrete subset of $\mathbb{R}$. In particular it holds that $\tilde{h}(1) = \tilde{h}(1/2) + q/2 = \tilde{h}(0) + q$. Note that q represents the number of times that the image of $\tilde{h}$ turns around S^1, i.e., $g_*([\gamma]) \in \pi_1(S^1)$ is q times a generator of $\pi_1(S^1)$. In particular, $g_*([\gamma]) \in \pi_1(S^1)$ is non-zero, because q is odd. However, this contradicts the fact that $[\gamma]$ is the identity in $\pi_1(S^2)$, since $\pi_1(S^2)$ is trivial.

Solution of 12.8: We prove the result for $k = 3$, which obviously implies the statement for $k = 1, 2$. Assume by contradiction that there exist closed subsets $A_1, A_2, A_3 \subseteq S^2$ that cover S^2 and such that A_i do not contain any pair of antipodal points for $i = 1, 2, 3$. For every $i = 1, 2, 3$, define $B_i := -A_i = \{-x \in S^2 : x \in A_i\}$. Then consider the continuous function $f_i\colon S^2 \to \mathbb{R}$ given by $f_i(x) := d(x, B_i) - d(x, A_i)$, where d is the standard distance on S^2. Note that, for all $x \in A_i$, we have $f_i(x) = d(x, B_i) > 0 > -d(-x, A_i) = f_i(-x)$, since $A_i \cap B_i = \emptyset$. In particular $f_i(x) \neq f_i(-x)$ for all $x \in A_i$ or $-x \in A_i$.

Now let $f\colon S^2 \to \mathbb{R}^2$ be given as $f(x) := (f_1(x), f_2(x))$. The function f is continuous, hence there exists $x_0 \in S^2$ such that $f(x_0) = f(-x_0)$, by item (ii) of Problem 12.7. Therefore we have that $f_1(x_0) = f_1(-x_0)$ and $f_2(x_0) = f_2(-x_0)$. As a result, by the observation above, x_0 and $-x_0$ are not contained in $A_1 \cup A_2$ (because otherwise $f_i(x_0) \neq f_i(-x_0)$ for $i = 1$ or $i = 2$). But then x_0 and $-x_0$ are both contained in A_3, since $S^2 = A_1 \cup A_2 \cup A_3$, which contradicts the assumption that A_3 do not contain any pair of antipodal points.

On the other hand, we prove that for every $k \geq 4$ there exist k closed subsets $A_1, \ldots, A_k$ that cover S^2 and such that A_i does not contain any pair of antipodal points for every $i = 1, \ldots, k$. It is sufficient to prove the result for $k = 4$, then for $k > 4$ the same statement trivially follows. Consider 4 points $p_1, p_2, p_3, p_4 \in S^2$ lying on the vertices of a tetrahedron inscribed in the sphere. In particular $d(p_i, p_j) = c < \pi$ for all $i \neq j$ and for some constant $c > 0$ (where d is the standard distance on S^2 as above). Moreover $\max_{i=1,2,3,4} d(x, p_i) \leq c$ for all $x \in S^2$. Then it is easy to check that the closed subsets A_1, A_2, A_3, A_4 defined as $A_i := \overline{B(p_i, c)} = \{x \in S^2 : d(x, p_i) \leq c\}$ satisfy the desired properties.

Solution of 12.9: We want to prove that the fundamental group of $\mathbb{P}^2(\mathbb{R})$ minus one point is $\mathbb{Z}$.

Proof via Van Kampen's Theorem. Let us represent the projective space $\mathbb{P}^2(\mathbb{R})$ as the quotient of the two-dimensional unit disk $D^2 \subseteq \mathbb{R}^3$ with the equivalent relation $\simeq$ identifying antipodal points on the boundary ∂D^2 (see Lecture 12 and Problem 7.7).

Without loss of generality we can assume the point to be the origin $(0,0)$ in $D^2/\simeq$. Observe that $D^2 \setminus \{(0,0)\}$ deformation retracts on its boundary via the homotopy

$$H(x,t) = x + t\left(\frac{1}{\|x\|} - 1\right)x.$$

Moreover note that $\partial D^2/\simeq$ is homeomorphic to a closed circle. Therefore we obtain that

$$\pi_1(\mathbb{P}^2(\mathbb{R}) \text{ minus a point}) = \pi_1((D^2/\simeq) \setminus \{(0,0)\}) = \pi_1(\partial D^2/\simeq) = \pi_1(S^1) = \mathbb{Z}.$$

Proof via covering arguments. Let us consider the double cover $p\colon S^2 \to \mathbb{P}^2(\mathbb{R})$ obtained identifying each pair of antipodal points. Moreover let us fix a point $u \in \mathbb{P}^2(\mathbb{R})$, so that we have to compute $\pi_1(\mathbb{P}^2(\mathbb{R}) \setminus \{u\})$. Note that $\{p^{-1}(u)\}$ consists of 2 points on S^2 and $q := p|_{S^2\setminus\{p^{-1}(u)\}}\colon (S^2 \setminus \{p^{-1}(u)\}) \to (\mathbb{P}^2(\mathbb{R}) \setminus \{u\})$ is a double cover. Without loss of generality we can assume that $\pi^{-1}(u) = \{N, S\}$ are the north and the south poles.

Since $S^2 \setminus \{N,S\}$ is homeomorphic to $\mathbb{R}^2$ minus one point, by Problem 10.9 we have that $\pi_1(S^2 \setminus \{N,S\}) = \mathbb{Z}$. Moreover we know that $q_*\colon \pi_1(S^2 \setminus \{N,S\}) \to \pi_1(\mathbb{P}^2(\mathbb{R}))$ is injective and the index of its image is equal to the number of sheets of the covering space, which is two. Let $\gamma_1\colon [0,1] \to S^2$ be a curve parametrizing half of the horizontal equatorial circle and let $\gamma_2\colon [0,1] \to S^2$ be the curve parametrizing the other half of the horizontal equatorial circle in such a way that $[\gamma_1 * \gamma_2]$ is a generator of $\pi_1(S^2 \setminus \{N,S\})$. Note that $\gamma := q \circ \gamma_1 = q \circ \gamma_2\colon [0,1] \to \mathbb{P}^2(\mathbb{R}) \setminus \{u\}$ is a closed curve in $\mathbb{P}^2(\mathbb{R}) \setminus \{u\}$ and $[\gamma] \in \pi_1(\mathbb{P}^2(\mathbb{R}) \setminus \{u\}) \setminus q_*(\pi_1(S^2 \setminus \{N,S\}))$.

As a result, since $q_*(\pi_1(S^2 \setminus \{p^{-1}(u)\})) < \pi_1(\mathbb{P}^2(\mathbb{R}))$ has index 2, $q_*(\pi_1(S^2 \setminus \{p^{-1}(u)\}))$ and $[\gamma]$ generate $\pi_1(\mathbb{P}^2(\mathbb{R}) \setminus \{u\})$. However, $[\gamma]^2 = [q \circ \gamma_1][q \circ \gamma_2] = [q \circ (\gamma_1 * \gamma_2)]$ generates $q_*(\pi_1(S^2 \setminus \{p^{-1}(u)\})) = \mathbb{Z}$. As a result, $[\gamma]$ generates $\pi_1(\mathbb{P}^2(\mathbb{R}) \setminus \{u\})$, which must then be equal to $\mathbb{Z}$.

Solution of 12.10: