3. Any path admits an injective sub-path

Since none of you were able to obtain a full solution to Challenge Problem 3, here is as note about a critical step of the proof you all have encountered (and which was indeed very difficult). *(Giada Franz)*

Proposition. *Let X be a Hausdorff topological space. Given any continuous curve $\gamma\colon [0,1] \to X$ with $\gamma(0) \neq \gamma(1)$, there exists an injective continuous curve $\tilde\gamma\colon [0,1] \to X$ with $\tilde\gamma(0) = \gamma(0)$, $\tilde\gamma(1) = \gamma(1)$ and $\tilde\gamma([0,1]) \subseteq \gamma([0,1])$.*

Proof. First, we define the curve $\beta\colon [0,1] \to X$ as $\beta(s) \coloneqq \gamma(t(s))$ for all $s \in [0,1]$, where

$$t(s) \coloneqq \inf A(s), \quad \text{where } A(s) \coloneqq \{t \in [0,s] \,:\, \exists\, t' \geq s \text{ s.t. } \gamma(t') = \gamma(t)\}.$$

Observe that $s \in A(s)$ and $A(s)$ is closed for every $s \in [0,1]$. Therefore $t(s)$ is well-defined and actually it holds $t(s) = \min A(s)$. Moreover $\beta([0,1]) \subseteq \gamma([0,1])$ and $\beta(0) = \gamma(0)$, $\beta(1) = \gamma(t(1)) = \gamma(1) = 1$.

Claim 1. *$t : [0,1] \to [0,1]$ is monotonically non-descreasing.*

Proof. Given $0 \leq s_1 < s_2 \leq 1$, we want to prove that $t(s_1) \leq t(s_2)$. If $t(s_2) > s_1 \geq t(s_1)$, there is nothing to prove; thus assume $t(s_2) \leq s_1$. Hence

$$\begin{aligned} t(s_2) &= \inf\{t \in [0,s_1] \,:\, \exists\, t' \geq s_2 \text{ s.t. } \gamma(t') = \gamma(t)\} \\ &\geq \inf\{t \in [0,s_1] \,:\, \exists\, t' \geq s_1 \text{ s.t. } \gamma(t') = \gamma(t)\} = t(s_1), \end{aligned}$$

as we wanted. □

Claim 2. *$t : [0,1] \to [0,1]$ is left-continuous, i.e. $\lim_{s\to s_0^-} t(s) = t(s_0)$ for all $s_0 \in [0,1]$.*

Proof. Let $\{s_n\}_{n\in\mathbb{N}} \subseteq [0,1]$ be an increasing sequence converging to some $s \in [0,1]$. We want to show that $t(s_n) \to t(s)$ as $n \to \infty$ (which is sufficient to prove the claim). First observe that $\{t(s_n)\}_{n\in\mathbb{N}}$ is non-decreasing by Claim 1 and $t(s_n) \leq t(s)$ for all $n \in \mathbb{N}$. Hence $t(s) \geq t_0 \coloneqq \lim_{n\to\infty} t(s_n) = \sup_{n\in\mathbb{N}} t(s_n)$. On the other hand, by definition of $t(s_n)$, for all $n \in \mathbb{N}$ there exists $t'_n \geq s_n$ such that $\gamma(t'_n) = \gamma(t(s_n))$. Then, up to subsequence, $\{t'_n\}_{n\in\mathbb{N}}$ converges to some $t' \geq \sup_{n\in\mathbb{N}} s_n = s$. By continuity of γ, this proves that $\gamma(t') = \gamma(t_0)$. Hence $t_0 \in A(s)$, which proves that $t(s) \leq t_0 = \lim_{n\to\infty} t(s_n)$. This, combined with the inequality above, proves that $t(s) = \lim_{n\to\infty} t(s_n)$ as we wanted. □

Claim 3. *$t : [0,1] \to [0,1]$ is "almost right-continuous", i.e. either $\lim_{s\to s_0^+} t(s) = t(s_0)$ or $\gamma(t(s_0)) = \gamma(s_0)$ for all $s_0 \in [0,1]$.*

Proof. Fix $s_0 \in [0,1]$ and assume that $\gamma(t(s_0)) \neq \gamma(s_0)$. Then there exists $t' > s_0$ such that $\gamma(t(s_0)) = \gamma(t')$. Now consider $s_0 \leq s \leq t'$. By Claim 1, $t(s_0) \leq t(s)$. On the other hand, $t(s_0) \in A(s)$, hence $t(s) = t(s_0)$. This proves that there exists $\varepsilon > 0$ such that $t(s) = t(s_0)$ for all $s_0 \leq s \leq s + \varepsilon$, in particular t is "almost right-continuous". □

Claim 4. *β is "almost injective", i.e. if $\beta(s_1) = \beta(s_2)$ for some $0 \leq s_1 \leq s_2 \leq 1$, then $\beta(s) = \beta(s_1) = \beta(s_2)$ for all $s \in [s_1, s_2]$.*

Proof. Consider $0 \le s_1 \le s_2 \le 1$ such that $\beta(s_1) = \beta(s_2)$, that is $\gamma(t(s_1)) = \gamma(t(s_2))$, and fix $s \in [s_1, s_2]$. By definition of $t(s_2)$, there exists $t' \ge s_2 \ge s$ such that $\gamma(t') = \gamma(t(s_2)) = \gamma(t(s_1))$; hence $t(s_1) \in A(s)$ (since $t(s_1) \le s_1 \le s$), which implies $t(s) \le t(s_1)$. This, together with Claim 1, proves that $t(s) = t(s_1)$ for all $s \in [s_1, s_2]$ and thus $\beta(s) = \gamma(t(s)) = \gamma(t(s_1)) = \beta(s_1)$. □

Claim 5. *β is continuous.*

Proof. Fix any $s_0 \in [0,1]$, then

$$\lim_{s\to s_0^-} \beta(s) = \lim_{s\to s_0^-} \gamma(t(s)) \overset{\bigstar}{=} \gamma(\lim_{s\to s_0-} t(s)) = \gamma(t(s_0)) = \beta(s_0).$$

Observe that equality ★ follows from the hypothesis of X being Hausdorff. Analogously, if we fall in the first case of Claim 3, we have that $\lim_{s\to s_0^+} \beta(s) = \beta(s_0)$. Hence, let us assume that $\gamma(t(s_0)) = \gamma(s_0)$. By Claim 4, $\gamma(s) = \gamma(t(s_0)) = \gamma(s_0)$ for all $s \in [t(s_0), s_0]$. Hence, if $t(s_0) < s_0$, we easily obtain that β is right-continuous at s_0 in this case too. Viceversa, suppose $t(s_0) = s_0$. Then observe that, for all $s \ge s_0$, $s_0 = t(s_0) \le t(s) \le s$. Hence $\lim_{s\to s_0^+} t(s) = t(s_0)$ and again we obtain the right-continuity of β at s_0. Therefore, we proved that β is left- and right-continuous at every $s_0 \in [0,1]$ and thus β is continuous. □

Consider on $[0,1]$ the equivalent relation given by $s_1 \sim s_2$ if and only if $\beta(s_1) = \beta(s_2)$. Then define $J \coloneqq [0,1]/\sim$ with quotient map $\pi\colon [0,1] \to J$ and $\tilde{\gamma}\colon J \to X$ such that $\gamma = \beta \circ \pi^{-1}$, which is well-defined since β is constant in the counterimage of a point through π. Moreover, $\tilde{\gamma}$ is patently injective connecting $\gamma(0)$ and $\gamma(1)$ and it is easily seen to be continuous. Therefore, the proposition we want to prove follows if we show that J is homeomorphic to $[0,1]$.

Thanks to Claims 4 and 5, $\pi^{-1}(r) \subseteq [0,1]$ is a closed interval for all $r \in J$. In particular, for all $x \in [0,1]$, we can define $[a_x, b_x] \coloneqq \pi^{-1}(\pi(x))$. Note that we could possibly have $a_x = b_x$. Moreover observe that the relation $\sim$ can be equivalently defined as $s_1 \sim s_2$ if and only if $s_1, s_2 \in [a_x, b_x]$ for some $x \in [0,1]$.

We now define a function $h\colon [0,1] \to [0,1]$ as follows. First we set $h(x) = 0$ for all $x \in [a_0, b_0]$ and $h(x) = 1$ for all $x \in [a_1, b_1]$. Then we are left with the open interval (b_0, a_1) to be mapped in $(0,1)$. In a general step, if we have an open interval $I = (u,v) \subseteq [0,1]$ to be mapped in an open interval $I' = (u', v') \subseteq [0,1]$, we define $m \coloneqq (u+v)/2$, $m' \coloneqq (u'+v')/2$ and then we set $h(x) = m'$ for all $x \in [a_m, b_m]$ and we recur on the intervals (u, a_m), (b_m, v) to be mapped in (u', m'), (m', v') respectively. Therefore, at each step $k \in \mathbb{N}$ of this process, h is defined in a union of closed intervals $V_k \subseteq [0,1]$ and it is monotonically non-decreasing on these intervals, with image $V_k \coloneqq h(U_k)$. Moreover note that $\bigcup_{k\in\mathbb{N}} V_k$ and $\bigcup_{k\in\mathbb{N}} V_k'$ are dense in $[0,1]$. Hence h extends in a unique way to a continuous non-decreasing bijective function $h\colon [0,1] \to [0,1]$.

Observe that h descends to the quotient $J = [0,1]/\sim$ as $\tilde{h}\colon J \to I$, since $h(s_1) = h(s_2)$ if and only if $s_1, s_2 \in [a_x, b_x]$ for some $x \in [0,1]$. Moreover $\tilde{h}$ is continuous and bijective, hence it is a homeomorphism since J and $[0,1]$ are compact. This shows that J and $[0,1]$ are homeomorphic, as we wanted, and concludes the proof. □