Eidgenössische Technische Hochschule Zürich
Swiss Federal Institute of Technology Zurich
Prof. Dr. Alessandro Carlotto 15 June 2020

Topology
Probeprüfung

Identifying number:	0	0	1

- Put your student identity card visible onto the desk.
- The exam is pseudonymised, following D-MATH recommendation. This means that you do **not** have to write your name or your student number on the exam, but you are identified by the number we told you prior to the exam. Please check the correctness of the identifying number at the top of this page!
- During the exam no written aids nor calculators or any other electronic device are allowed. **Phones must be switched off and stowed away** in your bag during the whole duration of the exam.
- A4-paper is provided. No other paper is allowed. Write with blue or black pens. **Do *not* use pencils, erasable pens, red or green ink, nor Tipp-Ex.**
- Write your solution to each exercise in the designated white space in the booklet. Leave enough ($\approx 2\,\text{cm}$) empty space on the margins (top, bottom and sides). If you need more space to write your solutions you can use the provided A4-paper, but please start every part on a new sheet of paper and **write your identifying number on every sheet of paper**.
- You will be asked to return **all** sheets of paper you are assigned, however you have the freedom to clearly cross those sheets you do not want us to consider during the grading process (i.e. scratch paper).
- Please write neatly! Please do not put the graders in the unpleasant situation of being incapable of reading your solutions, as this will certainly not play in your favour!
- All your answers do need to be properly justified (see in particular Part Ib). It is fine and allowed to use theorems/statements proved in class or in the homework (i. e. in Problem Sets 1–12) without reproving them (**unless otherwise stated**), but you should provide a precise statement of the result in question.
- Said $0 \leq x \leq 70$ your total score on the exercises, your final grade in the exam will be *bounded from below* by

$$\min\left\{6, \frac{x}{10}\right\}.$$

- The duration of the exam is **120 minutes**.

Do not fill out this table!

part	points	check
Ia	[10]	
Ib	[20]	
IIa	[20]	
IIb	[20]	
total	[70]	

grade:

Part Ia **[10 points]**

1. Let X be a path-connected topological space and fix $x_0 \in X$. Define the fundamental group $\pi_1(X, x_0)$. If $x_0, x_1 \in X$, what is the relation between $\pi_1(X, x_0)$ and $\pi_1(X, x_1)$?

2. What does it mean to say that a topological space X is semilocally simply connected? When did we encounter this assumption?

3. Define the Alexandroff one-point compactification of a topological space.

Part Ib [20 points]

Achtung! Unjustified answers do not give any point!

1. Every topology on a finite set has an even number of open sets.

☐ True ☐ False

2. If a topological space satisfies the second countability axiom then it satisfies the first countability axiom as well.

☐ True ☐ False

3. There exist topologies on the real line that make it compact.

☐ True ☐ False

4. Let $f\colon X \to Y$ be an open map and let $D \subseteq Y$ be a dense subset of Y. Then $f^{-1}(D)$ is dense in X.

☐ True ☐ False

5. Any continuous map $f\colon S^2 \to T^2$ admits a lift $\tilde{f}\colon S^2 \to \mathbb{R}^2$ with respect to the standard projection $p\colon \mathbb{R}^2 \to T^2$.

☐ True ☐ False

6. A quotient map $f\colon X \to Y$ is open if and only if it is a homeomorphism.

☐ True ☐ False

7. The unit sphere of $L^2(0,1)$, namely

$$X := \left\{ u \in L^2(0,1) \ : \ \int_0^1 u^2 = 1 \right\},$$

is separable.

☐ True ☐ False

8. There exists a homeomorphism from the Cantor set to a proper subset of the Cantor set.

☐ True ☐ False

9. Consider the collection $\mathcal{C}$ of the ten topological spaces, given by

$$0 \quad 1 \quad 2 \quad 3 \quad 4 \quad 5 \quad 6 \quad 7 \quad 8 \quad 9$$

each of them being regarded as a subspace of $\mathbb{R}^2$. Let $\simeq$ denote the homotopical equivalence in $\mathcal{C}$. Then $\mathcal{C}/\simeq$ contains exactly 5 elements.

☐ True ☐ False

10. In the setting of the previous question, let now $\cong$ denote homeomorphic equivalence in $\mathcal{C}$. Then $\mathcal{C}/\cong$ contains exactly 7 elements.

☐ True ☐ False

Part IIa [20 points]

The Sorgenfrey line $\mathbb{R}_{Sf}$ is the topological space obtain from $\mathbb{R}$ equipped with the topology generated by the basis

$$\mathcal{B} := \{[a, b) \, : \, a, b \in \mathbb{R}\}.$$

(i) Check that $\mathcal{B}$ is indeed a basis of a topology.

(ii) Prove that $\mathbb{R}_{Sf}$ is first countable.

(iii) Prove that $\mathbb{R}_{Sf}$ is Hausdorff.

(iv) Prove that $\mathbb{R}_{Sf}$ is separable.

(v) Prove that $A := \{(x, y) \in \mathbb{R}_{Sf} \times \mathbb{R}_{Sf} \, : \, x + y = 0\}$ is discrete in $\mathbb{R}_{Sf} \times \mathbb{R}_{Sf}$, in the sense that for all $(x, y) \in \mathbb{R}_{Sf} \times \mathbb{R}_{Sf}$ there exists an open neighborhood $U \subseteq \mathbb{R}_{Sf} \times \mathbb{R}_{Sf}$ of (x, y) such that $(U \setminus \{(x, y)\}) \cap A = \emptyset$.

(vi) Prove that $\mathbb{R}_{Sf}$ is not second countable.

(vii) Prove that $\mathbb{R}_{Sf}$ is not metrisable.

Part IIb [20 points]

For $p \geq 2$, set $\omega_p = e^{2\pi i/p}$ (so that ω_p is a p-th root of unity in $\mathbb{C}$). Consider the equivalence relation on $S^3 \subset \mathbb{C}^2$ given by

$$(z, w) \sim (z', w') \quad \text{if and only if} \quad \begin{cases} (z, w) = (z', w') \\ \text{or } z = \omega_p^k z' \text{ and } w = \omega_p^k w' \text{ for some } k \in \mathbb{Z}, \end{cases}$$

and consider the topological space $X_p := S^3/\sim$, endowed with the quotient topology; let $\pi\colon S^3 \to X_p$ be the corresponding projection.

(i) Prove that π is actually a covering map, and determine its degree.

(ii) Prove that X_p is a *path-connected* topological manifold.

(iii) Compute the fundamental group of X_p.

Let us now define the topological space $Y := S^3 \setminus (S^1 \times \{0\})$, obtain as S^3 minus a circle. Consider on Y the same equivalence relation $\sim$ as above and define $Y_p := Y/\sim$.

(iv) Compute the fundamental group of Y_p.