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Topology
Solutions - Probeprüfung

Part Ia [10 points]

1. Let X be a path-connected topological space and fix $x_0 \in X$. Define the fundamental group $\pi_1(X, x_0)$. If $x_0, x_1 \in X$, what is the relation between $\pi_1(X, x_0)$ and $\pi_1(X, x_1)$?

Let $\Omega(X, x_0)$ be the space of loops based on x_0, where we consider the equivalence relation $f \simeq f'$ for $f, f' \in \Omega(X, x_0)$ if and only if f and f' are homotopic. Then the fundamental group $\pi_1(X, x_0)$ is defined as the quotient $\Omega(X, x_0)/\simeq$ with the operation given by concatenation of loops.

Given $x_0, x_1 \in X$, let $h\colon [0,1] \to X$ be a path between x_0 and x_1. Then the groups $\pi_1(X, x_0)$ and $\pi_1(X, x_1)$ are isomorphic via the map $[f] \mapsto [h * f * h^{-1}]$.

2. What does it mean to say that a topological space X is semilocally simply connected? When did we encounter this assumption?

A topological space X is semilocally simply connected if for every $x \in X$ there exists a path-connected neighborhood U of x such that the inclusion map $\pi_1(U, x) \to \pi_1(X, x)$ is trivial.

This is a necessary condition in order to admit a universal cover. More in general, we encountered this assumption when we proved the existence of covers with assigned fundamental group.

3. Define the Alexandroff one-point compactification of a topological space.

Let (X, τ) be any topological space. Define the set $X^* := X \cup \{\infty\}$, which is the disjoint union of X with one point, and consider the family τ^* of subsets of X^* given by

$$\tau^* := \{U \subseteq X \,:\, U \in \tau\} \cup \{X^* \setminus C \,:\, C \subseteq X \text{ closed and compact}\}.$$

Then (X^*, τ^*) is the Alexandroff one-point compactification of (X, τ).

Note: Often it is required that the topological space (X, τ) is Hausdorff, locally compact, and non-compact. In this way the Alexandroff one-point compactification X^* is compact, Hausdorff, and such that X is dense in X^*.

Part Ib [20 points]

1. Every topology on a finite set has an even number of open sets.

☐ True ☒ False

For example, consider on the finite set $\{a, b\}$ with two elements the topology with open sets $\{\emptyset, \{a\}, \{a, b\}\}$.

2. If a topological space satisfies the second countability axiom then it satisfies the first countability axiom as well.

☒ True ☐ False

Assume that a topological space X satisfies the second countability axiom, i.e. there exists a countable basis $\mathcal{U}$ for its topology. Then, fixed $x \in X$, the set $\mathcal{U}_x := \{U \in \mathcal{U} : x \in U\}$ is a countable basis of neighborhoods for x. Hence X is first countable.

3. There exist topologies on the real line that make it compact.

☒ True ☐ False

Consider on $\mathbb{R}$ the trivial topology $\tau := \{\emptyset, \mathbb{R}\}$. Then $(\mathbb{R}, \tau)$ is obviously compact.

4. Let $f: X \to Y$ be an open map and let $D \subseteq Y$ be a dense subset of Y. Then $f^{-1}(D)$ is dense in X.

☒ True ☐ False

Let $U \subseteq X$ any open set. Since f is open, $f(U)$ is open in Y and thus there exists $y \in f(U) \cap D$. Then observe that there is $x \in U$ such that $f(x) = y$, which implies that $x \in f^{-1}(D) \cap U$. By arbitrariness of U, this proves that $f^{-1}(D)$ is dense in X.

5. Any continuous map $f: S^2 \to T^2$ admits a lift $\tilde{f}: S^2 \to \mathbb{R}^2$ with respect to the standard projection $p: \mathbb{R}^2 \to T^2$.

☒ True ☐ False

Since S^2 is simply connected, it holds that $f_*(\pi_1(S^2)) = p_*(\pi_1(\mathbb{R}^2)) = 0 \in \pi_1(T^2)$. Therefore, by the lifting criterion, there exists a lift $\tilde{f}: S^2 \to \mathbb{R}^2$ of f.

6. A quotient map $f\colon X \to Y$ is open if and only if it is a homeomorphism.

☐ True ☒ False

Let us consider $X = \mathbb{R}$, $Y = \mathbb{R}/\mathbb{Z}$ and $f\colon \mathbb{R} \to \mathbb{R}/\mathbb{Z}$ the quotient map. Then f is open (for example because f is a covering), but it is patently not a homeomorphism.

7. The unit sphere of $L^2(0,1)$, namely

$$X := \left\{ u \in L^2(0,1) \ : \ \int_0^1 u^2 = 1 \right\},$$

is separable.

☒ True ☐ False

First recall that $L^2(0,1)$ is separable (a countable dense subset being given by simple functions with rational coefficients on intervals with rational endpoints). As a result, X is separable as well since it is a subspace of a separable metric space.

8. There exists a homeomorphism from the Cantor set to a proper subset of the Cantor set.

☒ True ☐ False

Let $C \subseteq [0,1]$ be the Cantor set and consider the map $f\colon C \to C$ given by $f(x) = x/3$. Then f is a homeomorphism between C and $C \cap [0,1/3]$.

9. Consider the collection $\mathcal{C}$ of the ten topological spaces, given by

0 1 2 3 4 5 6 7 8 9

each of them being regarded as a subspace of $\mathbb{R}^2$. Let $\simeq$ denote the homotopical equivalence in $\mathcal{C}$. Then $\mathcal{C}/\simeq$ contains exactly 5 elements.

☐ True ☒ False

The space $\mathcal{C}/\simeq$ contains 3 elements. Indeed $1,2,3,5,7$ deformation retract to a point, thus they are all homotopical equivalent to a point. The elements $0,4,6,9$ deformation retract to a circle, thus they are all homotopical equivalent to a circle. Finally 8 is the wedge sum of two circles. As a result, $\mathcal{C}$ contains three classes of equivalence with respect to $\simeq$.

10. In the setting of the previous question, let now $\cong$ denote homeomorphic equivalence in $\mathcal{C}$. Then $\mathcal{C}/\cong$ contains exactly 7 elements.

☐ True ☒ False

Note that $2, 3, 5, 7$ are homeomorphic to a segment and 6 is homeomorphic to 9 (they are obtained one from the other by rotating of 180°). As a result, $\mathcal{C}/\cong$ contains at most 6 elements, i.e., $[0]$, $[1]$, $[2] = [3] = [5] = [7]$, $[4]$, $[6] = [9]$, $[8]$. Indeed one can check that it contains exactly 6 elements.

Part IIa [20 points]

The Sorgenfrey line $\mathbb{R}_{Sf}$ is the topological space obtain from $\mathbb{R}$ equipped with the topology generated by the basis

$$\mathcal{B} := \{[a,b) \, : \, a,b \in \mathbb{R}\}.$$

(i) Check that $\mathcal{B}$ is indeed a basis of a topology.

(ii) Prove that $\mathbb{R}_{Sf}$ is first countable.

(iii) Prove that $\mathbb{R}_{Sf}$ is Hausdorff.

(iv) Prove that $\mathbb{R}_{Sf}$ is separable.

(v) Prove that $A := \{(x,y) \in \mathbb{R}_{Sf} \times \mathbb{R}_{Sf} \, : \, x + y = 0\}$ is discrete in $\mathbb{R}_{Sf} \times \mathbb{R}_{Sf}$, in the sense that for all $(x,y) \in \mathbb{R}_{Sf} \times \mathbb{R}_{Sf}$ there exists an open neighborhood $U \subseteq \mathbb{R}_{Sf} \times \mathbb{R}_{Sf}$ of (x,y) such that $(U \setminus \{(x,y)\}) \cap A = \emptyset$.

(vi) Prove that $\mathbb{R}_{Sf}$ is not second countable.

(vii) Prove that $\mathbb{R}_{Sf}$ is not metrisable.

(i) First note that $\mathcal{B}$ covers $\mathbb{R}$. Now consider any $[a,b),[c,d) \in \mathcal{B}$, then $[a,b) \cap [c,d) = [\max(a,b), \min(c,d)) \in \mathcal{B}$. Hence $\mathcal{B}$ is a basis of a topology on $\mathbb{R}$.

(ii) Given any $x \in \mathbb{R}_{Sf}$, consider the subfamily $\mathcal{B}_x := \{[x, x+1/n) \, : \, n \in \mathbb{N}_*\}$ of $\mathcal{B}$. Note that for any $[a,b) \in \mathcal{B}$ that contains x there exists $n \in \mathbb{N}_*$ such that $[x, x+1/n) \subseteq [a,b)$. Therefore $\mathcal{B}_x$ is a countable basis of neighborhoods of x, which proves that $\mathbb{R}_{Sf}$ is first countable by arbitrariness of x.

(iii) Let $x, y \in \mathbb{R}_{Sf}$ be two distinct points. Without loss of generality we can assume that $x < y$. Then we have that $[x,y)$ and $[y, y+1)$ are two disjoint open sets in $\mathbb{R}_{Sf}$ containing x and y respectively. This proves that $\mathbb{R}_{Sf}$ is Hausdorff since x and y are arbitrary.

(iv) Consider the countable subset $\mathbb{Q}$ of rational numbers in $\mathbb{R}_{Sf}$. We claim that $\mathbb{Q}$ is a dense subset of $\mathbb{R}_{Sf}$, which implies that $\mathbb{R}_{Sf}$ is separable. Consider any non-empty $[a,b) \in \mathcal{B}$, then there exists a rational number $p \in \mathbb{Q}$ such that $a \leq p < b$ since $\mathbb{Q}$ is dense in $\mathbb{R}$ with respect to the Euclidean topology. This implies that $\mathbb{Q}$ is dense in $\mathbb{R}_{Sf}$ by arbitrariness of $[a,b) \in \mathcal{B}$.

(v) Consider any $(x,y) \in \mathbb{R}_{Sf} \times \mathbb{R}_{Sf}$ and take $\varepsilon > 0$ such that

$$\varepsilon = \begin{cases} |x+y|/2 & \text{if } x+y \neq 0 \\ 1 & \text{if } x+y = 0. \end{cases}$$

We claim that $U := [x, x+\varepsilon) \times [y, y+\varepsilon)$ is an open neighborhood of (x,y) in $\mathbb{R}_{Sf} \times \mathbb{R}_{Sf}$ such that $(U \setminus \{(x,y)\}) \cap A = \emptyset$. The only non-trivial thing to check is that $(U \setminus \{(x,y)\}) \cap A = \emptyset$. Take any $(x+u, y+v) \in [x, x+\varepsilon) \times [y, y+\varepsilon)$ different from (x, y), i.e., $u + v > 0$. Then, if $x + y \neq 0$, we have

$$|(x + u) + (y + v)| \geq |x + y| - |u + v| > |x + y| - 2\varepsilon = 0,$$

and, if $x + y = 0$, we have $|(x + u) + (y + v)| = |u + v| > 0$. In both cases, $|(x + u) + (y + v)| > 0$ and thus $(x + u, y + v) \notin A$. This indeed proves that $(U \setminus \{(x, y)\}) \cap A = \emptyset$ as we wanted.

(vi) We first prove the following lemma.

Lemma. *Let X be a topological space that has a discrete uncountable subset A. Then X is not second countable.*

Proof. Let us consider any basis $\mathcal{O}$ for the topology of X. Moreover, for every $x \in A$, let $U_x \subseteq X$ be an open neighborhood of x such that $(U \setminus \{x\}) \cap A = \emptyset$, which exists because A is discrete. Since $\mathcal{O}$ is a basis, for every $x \in A$ there is $O_x \in \mathcal{O}$ such that $x \in O_x \subseteq U_x$. Observe that for every $x \neq y \in A$ we have that $O_x \neq O_y$, because $x \in O_x \setminus O_y$ and $y \in O_y \setminus O_x$. As a result $\{O_x\}_{x \in A}$ is an uncountable subset of $\mathcal{O}$, which is therefore uncountable. This proves that any basis of X is uncountable and thus that X is not second countable. □

Thanks to this lemma together with (v), we directly obtain that $\mathbb{R}_{Sf} \times \mathbb{R}_{Sf}$ is not second countable and this implies that $\mathbb{R}_{Sf}$ is not second countable as well. Indeed, if $\mathbb{R}_{Sf}$ would be secound countable, also $\mathbb{R}_{Sf} \times \mathbb{R}_{Sf}$ would be second countable, because the product of two second countable spaces is second countable.

(vii) This follows from (vi). Indeed, any separable metric space is second countable, but $\mathbb{R}_{Sf}$ is separable and not second countable, thus it cannot by metrisable.

Part IIb **[20 points]**

For $p \geq 2$, set $\omega_p = e^{2\pi i/p}$ (so that ω_p is a p-th root of unity in $\mathbb{C}$). Consider the equivalence relation on $S^3 \subset \mathbb{C}^2$ given by

$$(z,w) \sim (z',w') \quad \text{if and only if} \quad \begin{cases} (z,w) = (z',w') \\ \text{or } z = \omega_p^k z' \text{ and } w = \omega_p^k w' \text{ for some } k \in \mathbb{Z}, \end{cases}$$

and consider the topological space $X_p := S^3/\sim$, endowed with the quotient topology; let $\pi\colon S^3 \to X_p$ be the corresponding projection.

(i) Prove that π is actually a covering map, and determine its degree.

(ii) Prove that X_p is a *path-connected* topological manifold.

(iii) Compute the fundamental group of X_p.

Let us now define the topological space $Y := S^3 \setminus (S^1 \times \{0\})$, obtain as S^3 minus a circle. Consider on Y the same equivalence relation $\sim$ as above and define $Y_p := Y/\sim$.

(iv) Compute the fundamental group of Y_p.

(i) Let us consider the group $\mathbb{Z}/p\mathbb{Z}$ and the action $\varphi\colon G \to \mathrm{Homeo}(S^3)$ of $\mathbb{Z}/p\mathbb{Z}$ on S^3 given by

$$\varphi(k)\colon (z,w) \mapsto (\omega_p^k z, \omega_p^k w)$$

for all $k = 0, \dots, p-1 \in \mathbb{Z}/p\mathbb{Z}$. The map $\varphi(k)\colon S^3 \to S^3$ is a well-defined homeomorphism of S^3 and $\varphi\colon \mathbb{Z}/p\mathbb{Z} \to S^3$ is injective. In fact $\varphi(k)$ is an isometry of S^3 for all $k \in \mathbb{Z}/p\mathbb{Z}$.

Let us now check that the action $\varphi\colon \mathbb{Z}/p\mathbb{Z} \to \mathrm{Homeo}(S^3)$ is properly discontinuous. First observe that, if $\varphi(k)(z,w) = \varphi(h)(z,w)$ for some $k, h \in \mathbb{Z}/p\mathbb{Z}$ and $(z,w) \in S^3$, we get

$$(\omega_p^k z, \omega_p^k w) = (\omega_p^h z, \omega_p^h w) \iff (\omega_p^k - \omega_p^h)(z,w) = 0 \iff \omega_p^k = \omega_p^h \iff p \mid k-h.$$

Since $0 \leq k, h < p$, this implies that $k = h$. Therefore $\{\varphi(k)(z,w)\}_{k \in \mathbb{Z}/p\mathbb{Z}} = \{\omega_p^k (z,w)\}_{k=0,\dots,p-1}$ consists of distinct elements for every $(z,w) \in S^3$. Now, given any $(z,w) \in S^3$, take a sufficiently small neighborhood U of (z,w) (for example a ball $B_\varepsilon((z,w)) \subseteq S^3$ with $\varepsilon > 0$ sufficiently small). Then, using the fact that $\varphi(k)$ is an isometry for all $k \in \mathbb{Z}/p\mathbb{Z}$ and that $\{\varphi(k)(z,w)\}_{k \in \mathbb{Z}/p\mathbb{Z}}$ consists of distinct elements, we obtain that $\{\varphi(k)(U)\}_{k \in \mathbb{Z}/p\mathbb{Z}}$ consists of pairwise disjoint sets, which proves that $\mathbb{Z}/p\mathbb{Z}$ acts properly discontinuously on S^3 via φ.

At this point note that the equivalence relation $\sim$ is exactly defined as

$$(z,w) \sim (z',w') \quad \text{if and only if} \quad \begin{cases} (z,w) = (z',w') \\ \text{or } \exists\, k \in \mathbb{Z}/p\mathbb{Z} \text{ with } (z,w) = \varphi(k)(z',w'). \end{cases}$$

Therefore $X_p = S^3/(\mathbb{Z}/p\mathbb{Z})$ and we saw in class (Lecture 28) that $\pi\colon S^3 \to S^3/(\mathbb{Z}/p\mathbb{Z}) = X_p$ is a covering map. The degree of the covering is the cardinality of $\pi^{-1}([(z,w)]) = \{\varphi(k)(z,w)\}_{k\in\mathbb{Z}/p\mathbb{Z}}$, which is p (we already observed that $\{\varphi(k)(z,w)\}_{k\in\mathbb{Z}/p\mathbb{Z}}$ consists of p distinct elements).

(ii) First note that X_p is path-connected because it is image of the path-connected space S^3 via a surjective continuous map. Let us prove that X_p is a topological manifold. Let $[(z,w)]$ be any point in X_p and let U be an evenly covered open neighborhood of $[(z,w)]$. Then there exists an open neighborhood $\tilde{U}$ of $(z,w) \in S^3$ such that $\varphi\colon \tilde{U} \to U$ is a homeomorphism. Recall that S^3 is a topological manifold, hence there is an open neighborhood $\tilde{V} \subseteq \tilde{U}$ of (z,w) that is homeomorphic to $\mathbb{R}^3$. As a result $\varphi(\tilde{V}) \subseteq U$ is an open neighborhood of $[(z,w)] \in X_p$ homeomorphic to $\mathbb{R}^3$, as we wanted.

(iii) Since S^3 is simply connected, we saw in class that

$$\pi_1(X_p) = \pi_1(S^3/(\mathbb{Z}/p\mathbb{Z})) \cong \mathbb{Z}/p\mathbb{Z}.$$

(iv) Let us first prove that Y_p is homotopy equivalent to $(\mathbb{C}\times\mathbb{C}^*)/\!\sim$, where $\sim$ on $\mathbb{C}\times\mathbb{C}^*$ is defined in the same way as above, i.e., $(z,w) \sim (z',w')$ if and only if $(z,w) = (z',w')$ or $(z,w) = (\omega_p^k z', \omega_p^k w')$ for some $k \in \mathbb{Z}$.

For notational convenience, let us define $Z := \mathbb{C}\times\mathbb{C}^*$, $Z_p := (\mathbb{C}\times\mathbb{C}^*)/\!\sim$ and denote by $\pi\colon Z \to Z_p$ the quotient map. Consider the deformation retraction $H\colon Z\times[0,1] \to Y$ given by

$$H((z,w),t) = (1-t)(z,w) + t\frac{(z,w)}{\|(z,w)\|}.$$

Note that H descends to a deformation retraction $\tilde{H}\colon Z_p \times [0,1] \to Y_p$ between the quotients. As a result we get that Z_p and Y_p are homotopy equivalent. In particular $\pi_1(Y_p) = \pi_1(Z_p)$.

Now let us consider the universal cover $\pi'\colon \mathbb{C}\times\mathbb{C} \to Z$ defined as $(z,w) \mapsto (z, e^w)$. Note that $\pi\circ\pi'\colon \mathbb{C}\times\mathbb{C} \to Z_p$ is a universal cover of Z_p. Define the group action $\psi\colon \mathbb{Z} \to \mathrm{Homeo}(\mathbb{C}\times\mathbb{C})$ on $\mathbb{C}\times\mathbb{C}$ given by

$$\psi(k)\colon (z,w) \mapsto \left(e^{2\pi ik/p}z, w + \frac{2\pi ik}{p}\right) = \left(\omega_p^k z, w + \frac{2\pi ik}{p}\right).$$

First note that $\psi(\mathbb{Z}) < \mathrm{Homeo}(\mathbb{C}\times\mathbb{C})$ is isomorphic to $\mathbb{Z}$. Indeed ψ is obviously injective. Moreover it holds that $\psi(\mathbb{Z})$ is the group of deck transformations of the universal cover $\pi\circ\pi'\colon \mathbb{C}\times\mathbb{C} \to Z_p$. As a result, we obtain that $\pi_1(Z_p)$ is isomorphic to $\psi(\mathbb{Z}) \cong \mathbb{Z}$.