I would like to discuss two (a posteriori *fully equivalent*) perspectives one can take when introducing the notions of **interior, closure and boundary** of a set. The second approach is the one adopted by Hatcher. In some sense, it is closer to the perspective you may have adopted in earlier courses, but it has the disadvantage of making certain arguments somewhat heavier.

Anyways, it is better to keep both of them in mind, so to employ whichever is more convenient. Here is the logical skeleton of the lecture I gave today.

First approach.

Let X be a set and $\mathcal{O}$ be a topology on X. Let then $\mathcal{C}$ denote the closed sets with respect to this topology. Given $A \subset X$, we define:

$$\underbrace{\operatorname{int}(A) = \bigcup_{O\subset A, O\in\mathcal{O}} O}_{\text{interior}}, \qquad \underbrace{\overline{A} = \bigcap_{K\supset A, K\in\mathcal{C}} K}_{\text{closure}}, \qquad \underbrace{\partial A = \overline{A}\setminus \operatorname{int}(A)}_{\text{boundary}}.$$

Remark 1 *Trivialities: in the setting above, for any set A the following assertions hold true*

- *$\operatorname{int}(A)$ is open, and $\operatorname{int}(A) \subset A$;*
- *$\overline{A}$ is closed, and $\overline{A} \supset A$;*
- *∂A is closed.*

Proposition 2 *In the setting above, one has:*

1. *$A = \operatorname{int}(A)$ if and only if $A \in \mathcal{O}$;*
2. *$A = \overline{A}$ if and only if $A \in \mathcal{C}$.*

With this approach, the proof of the previous proposition is totally straightforward (we did it in class but try it yourself as a reality check).

Second approach.

Given $(X, \mathcal{O})$ a topological space, for any $x \in X$ one and only one of the following three (mutually exclusive) scenarios can occur:

1. there exists an open set $O \ni x, O \subset A$;
2. there exists an open set $O \ni x, O \subset X \setminus A$;
3. for any open set $O \ni x$ one has $O \cap A \neq \emptyset$ and $O \cap (X \setminus A) \neq \emptyset$.

That being said, one defines:

$$(\mathrm{int}(A))_H = \text{points for which 1. occurs}, \quad (\partial A)_H = \text{points for which 3. occurs},$$

and then $(\overline{A})_H = (\mathrm{int}(A))_H \cup (\partial A)_H$.

Remark 3 *Trivialities: in the setting above, for any set A the following assertion holds true*

$$X = (int(A))_H \sqcup (\partial A)_H \sqcup (int(X \setminus A))_H$$

*where $\sqcup$ denotes a **disjoint** union. In other words: any set A induces a partition of X in the three pieces above.*

Equivalence and implications.

Proposition 4 *In the setting above, one has for any set $A \subset X$ that*

$$int(A) = (int(A))_H, \quad \overline{A} = (\overline{A})_H, \quad \partial A = (\partial A)_H.$$

The proof amounts to simple set-theoretic manipulations (in class we have proven that $\overline{A} = (\overline{A})_H$, and the other two equalities are strictly simpler). So, in practice, there is no need of different notation and we can just write $\mathrm{int}(A)$, $\overline{A}$, ∂A for the interior, closure and boundary of a set, respectively.

Once this proposition is proven, note that we also derive (for free!) that

$$X = \mathrm{int}(A) \sqcup \partial A \sqcup \mathrm{int}(X \setminus A),$$

and this *induced partition* will often be helpful in the sequel of the course.