Exercise class 25/3

Topology FS 20

1 Convergence of sequences in arbitrary metric spaces / neighbourhood bases

Def: Let X be a topological space, $(x_n)_{n\geq 1} \subseteq X$ a sequence. We say that x_n <u>converges</u> to x (denoted $x_n \to x$) if for any $U \ni x$ open $\exists\, N \geq 1$ s.t. $\forall\, n \geq N$: $x_n \in U$.

When checking a topological property, one often wants to reduce checking for <u>all</u> open sets to only some "special" open sets. This is what the concept of basis does. Here we only care about what happens around x, which motivates the following definition:

Def: Let $x \in X$. We say that $(U_i)_{i\in I}$ ($\ni x$ open) is a <u>neighbourhood basis</u> of x if $\forall\, x \in U$ open $\exists\, i \in I$: $x \in U_i \subseteq U$.

Like a basis is enough to describe the topology, a neighbourhood basis is enough to describe the topology around a point. In particular:

Lemma: Let $(x_n)_{n\geq 1} \subseteq X$, $x \in X$, $(U_i)_{i\in I}$ a neighbourhood basis of x. Then $x_n \to x$ iff $\forall\ i \in I\ \exists\ N \geq 1$ s.t. $\forall\ n \geq N$: $x_n \in U_i$.
That is, we only need to check the condition at the U_i.

Pf. Given $x \in U$, choose i: $x \in U_i \subseteq U$. Then if $x_n \in U_i$ $\forall\ n \geq N$, the same holds for U_i. $\square$

EX: Let (M, d) be a metric space, $x \in M$. Then a neighbourhood basis of x is $\{B_r(x)\}_{r>0}$, by definition of the topology induced by d. Therefore, using the lemma above, the topological notion of convergence and the usual one coincide.

EX: Let X be a set equipped with the discrete topology. Then ~~a sequence~~ a neihbourhood basis of x is given by the single open set $\{x\}$. Therefore $x_n \to x$ iff $\exists\ N \geq 1$ s.t. $\forall\ n \geq N$: $x_n = x$. That is, a sequence converges iff it is eventually constant. ("$\Leftarrow$" is always true!)

- How about uniqueness of the limit?

EX: Let X be a set equipped with the trivial topology (only open sets are $\emptyset$ and X). Then ~~any~~ ~~sequen~~ a neighbourhood basis of x is $\{X\}$. Therefore any sequence converges to all points!

EX: For a more interesting example, let X be equipped with the cofinite topology (4.5). Let $(x_n)_{n\geq 1}$ be a sequence taking ~~[illegible]~~ ~~many~~ distinct values. Then ~~then~~ it converges to any $x \in X$. Indeed, given $x \in U$ open, for $N \geq 1$ large enough, $x_n \in U$ $\forall\, n \geq N$, since $X \setminus U$ is finite.

- What is the problem with these examples? They are not Hausdorff.

Def: X is *Hausdorff* if $\forall\, x \neq y$ $\exists\, U_x \ni x$, $U_y \ni y$ open s.t. $U_x \cap U_y = \emptyset$.

Prop°: Let X be Hausdorff, $(x_n)_{n\geq 1} \subseteq X$. Then $(x_n)_{n\geq 1}$ admits at most one limit.

Pf: Suppose that $x_n \to x$, $x_n \to y$, $x \neq y$. Take any $U_x \ni x$, $U_y \ni y$. Then for n large enough, $x_n \in U_x \cap U_y \neq \emptyset$. So X is not Hausdorff. ∎

- However, it is possible that a space has unique limits but it is not Hausdorff: we will see an example in a minute. (✱).

- Convergent sequences behave well wrt continuous maps:

Prop°: Let $f: X \to Y$ be continuous, and let $x_n \to x$. Then $f(x_n) \to f(x)$.

Pf: Let $U \ni f(x)$ be open in Y. Then $f^{-1}(U) \ni x$ is open in X. So $\exists\ N \geq 1$ s.t. $\forall\ n \geq N$: $x_n \in f^{-1}(U)$. So $\forall\ n \geq N$ $f(x_n) \in U$. ∎

- Is the converse true? NO! Let us now see a counterexample to this and (✱) ↑.

The cocountable topology

Def: Let X be a set. ~~Then~~ Let $\mathcal{T}_{coc} := \{\emptyset\} \cup \{Y \subseteq X : X \setminus Y \text{ is countable}\}$. $\mathcal{T}_{coc}$ is a topology, called the cocountable topology. It coincides with the discrete topology $\iff$ X is countable.

- That this is a topology is proven as in 4.5. Notice that if X is uncountable, any two nonempty open sets must intersect, so X is not Hausdorff.

∎

Lemma: Let X be equipped with the cocountable topology. Then a sequence converges iff it is eventually constant.

Rmk: The discrete topology also had this property, but this is a strictly coarser topology if X is uncountable.

Pf: Suppose $x_n \to x$. Let $C := (x_n)_{n \geq 1} \setminus \{x\}$ which is countable. So $x \in X \setminus C$ which is open. Thus $\exists\ N \geq 1$ s.t. $\forall\ n \geq N$: $x_n \in X \setminus C$. But this is only possible if $x_n = x$. ■

EX: Thus if X is uncountable with this topology, it is not Hausdorff, but limits are still unique.

EX: Let $f = id_X : (X, \tau_{coc}) \to (X, \tau_{disc})$, with X uncountable. This is clearly not continuous. However, if $x_n \to x$, then x_n is ev. const. $\Rightarrow f(x_n)$ is ev. const. $\Rightarrow f(x_n) \to f(x)$. (In fact, the latter is true for any f!)

– Moral of the story: there are weird spaces out there, and if we want to work with sequences just like in metric spaces, then ~~[illegible]~~ we need some extra assumption on our spaces.

2 First-countable spaces and convergence

Def: A space is <u>first-countable</u> if every point admits a countable neighbourhood basis.

EX: If (M, d) is a metric space, $\{B_r(x)\}_{r \in \mathbb{Q}_{>0}}$ is a countable neighbourhood basis of x.

EX: From the previous examples, the trivial and discrete topology are clearly first-countable.

Rmk: Most spaces you will encounter in life are first-countable.

– This turns out to be enough to get the converse of the previous proposition:

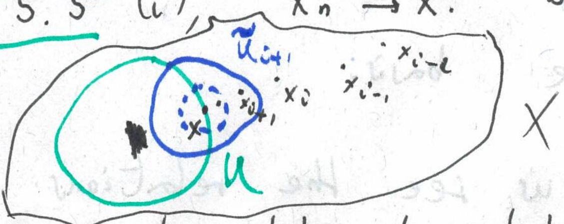
Prop°: Let $f: X \to Y$ be a map, X first-countable. Suppose that whenever $x_n \to x$, $f(x_n) \to f(x)$. Then f is continuous.

In order to prove this, we need the following lemma:

Lemma: Let X be a first-countable space. Then $U \subseteq X$ is open iff whenever $x_n \to x \in U$, $\exists$ $N \geq 1$ s.t. $\forall n \geq N$ $x_n \in U$.

LPf: "$\Rightarrow$" Follows from the definition of convergence (for any space).

"$\Leftarrow$" Suppose that U is not open, and let $x \in U \setminus \mathrm{int}(U)$. Let $(U_n)_{n\geq 1}$ be a neighbourhood basis of x, and let $\tilde{U}_n := \bigcap_{k=1}^{n} U_k$. Choose $x_n \in \tilde{U}_n \setminus U$, which exists because $x \notin \mathrm{int}(U)$. Then by ex. 5.5 (i), $x_n \to x$. But $x_n \notin U \ \forall\ n \geq 1$.

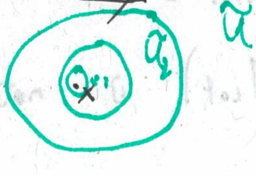


Proof of prop°: Let f be as in the statement. Let $U \subseteq Y$ be open, and let $x \in f^{-1}(U)$. Let $x_n \to x$. Then by hypothesis $f(x_n) \to f(x)$, so $\exists\ N \geq 1$ s.t. $\forall\ n \geq N$ $f(x_n) \in U$. So $\forall\ n \geq N$ $x_n \in f^{-1}(U)$. By the lemma, $f^{-1}(U)$ is open. So f is continuous.

Rmk: A space with this property is called sequential. This is a strictly weaker notion than first-countable, but let's not get into more weird examples.

- So in first-countable spaces convergent sequences determine the topology, just like in metric spaces, (which are first-countable).

EXO 5.5: The idea is that if $(U_n)_{n\geq 1}$ is a neighbourhood basis, then so is $\tilde{U}_n$ (since $x \in \tilde{U}_n \subseteq U_n$). But $\tilde{U}_n$ has the further property that if y is in $\tilde{U}_n$ then $y \in U_1, \dots, U_n$: if $y \in \tilde{U}_n$ for n large, then y is in many elements of a neighbourhood basis of x.

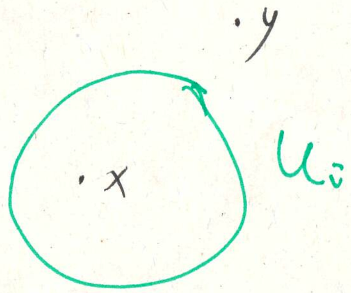


EXO 5.6 :

Lemma : Let X be a space in which all points are closed (this is called a T_1 space). Let $x \in X$, and let $(U_i)_{i \in I}$ be a neighbourhood basis of x. Then $\bigcap_{i \in I} U_i = \{x\}$.

Pf : Let $y \neq x$. Then $X \setminus \{y\}$ is open so $\exists\, U_i : x \in U_i \subseteq X \setminus \{y\}$.

So $y \notin U_i \Rightarrow y \notin \bigcap_{i \in I} U_i$.



- $(\mathbb{R}, \mathcal{T}_{cof})$ has this property. But can you obtain a point as the intersection of countably many cofinite sets ?

3 Countability axioms

– We have talked about first-countability already.

Def: A space is <u>separable</u> if it admits a countable dense subset. A space is <u>second-countable</u> if it admits a countable basis.

– Let us see the relations between these three axioms.

Lemma: A second-countable space is separable and first-countable.

Pf: Let $(B_n)_{n\geq 1}$ be a countable basis. Choose $x_n \in B_n$ for any $B_n \neq \emptyset$. Then $\{x_n\}_{n\geq 1}$ is dense. Indeed, let $x \in X$. Given $U \ni x$ open, choose $x \in B_n \subseteq U$. Then $x_n \in B_n \subseteq U$. So any neighbourhood of x intersects $\{x_n\}_{n\geq 1}$.

Let $x \in (B_n)_{n\geq 1}$. Let $I_x := \{ n \geq 1 : x \in B_n \}$. Then $(B_n)_{n\in I_x}$ is a neighbourhood basis of x.

EX: A first-countable space is not necessarily separable. For instance, consider $(\mathbb{R}, \mathcal{T}_{disc})$. This is first-countable, but any countable set is closed, so it cannot be dense.

EX: A separable space is not necessarily first-countable. For instance, $(\mathbb{R}, \mathcal{T}_{cof})$ is separable: the closure of any countably infinite set is $\mathbb{R}$, since the only other closed sets are finite. But by 5.6, $(\mathbb{R}, \mathcal{T}_{cof})$ is not first-countable.

- ~~Tho~~ These examples, together with the lemma, show also that separable $\not\Rightarrow$ second-countable, and first-countable $\not\Rightarrow$ $\not\Rightarrow$ second countable. What if we assume both?

~~Tho~~ Notice that a metric space which is separable is automatically ~~the~~ second-countable (see solution of 4.8 (ii)), so this seems promising. However, this is false for non-metric spaces.

The lower limit topology

Def: Let $\mathcal{B} := \{ [a,b) : a<b \} \subseteq \mathcal{P}(\mathbb{R})$. This is the basis for a topology $\mathcal{T}$, called the lower limit topology.

EX: $(\mathbb{R}, \mathcal{T})$ is separable: $\mathbb{Q}$ is dense. (any basic open sets contains a rational).

$(\mathbb{R}, \mathcal{T})$ is first-countable: a neighbourhood basis for x is $\{[x, x+q)\}_{q\in\mathbb{Q}_{>0}}$: if $x\in[a,b)$, then $\exists\, q\in\mathbb{Q}_{>0}$ s.t. $x\in[x,q)\subseteq[a,b)$.

$(\mathbb{R}, \mathcal{T})$ is not second-countable. Indeed, let $\mathcal{B}$ be a basis for $\mathcal{T}$. We define an injection of $\mathbb{R}$ in $\mathcal{B}$, showing that $\mathcal{B}$ must be uncountable. Given $x\in\mathbb{R}$, let $B_x\in\mathcal{B}$ be s.t. $x\in B_x\subseteq[x,x+1)$. Then $\min B_x = x$, so if $x\neq y$, then $B_x\neq B_y$. Thus $\mathbb{R}\to\mathcal{B}: x\mapsto B_x$ is injective.

EXO 5.9: Let X be a separable metric space. Then X is second-countable (solution of ex. 4.8 (ii)).

Lemma: Let X be second-countable, and $Y \subseteq X$ discrete. Then Y is countable.

Rmk: This actually holds ~~more~~ also ~~generally~~ for first-countable normal spaces, (harder, but you can try it!)

Pf: For any $y \in Y$, choose $B_y \in \mathcal{B}$ (a countable basis) s.t. $B_y \cap Y = \{y\}$ (Y is discrete). Then the map $Y \to \mathcal{B} : y \mapsto B_y$ is injective, so Y is countable. $\blacksquare$

Therefore if a metric space admits an uncountable discrete subspace, then it cannot be separable.

EXO 5.7: We have seen that first-countable + separable $\not\Rightarrow$ second-countable. However we can still define $\mathcal{P}$ to be the collection of all neighbourhood bases of the points in a countable dense subset. This is not necessarily a basis, but it still has some nice properties. For instance choosing a point in each set of $\mathcal{P}$ yields a countable dense subset (see the proof second-countable $\Rightarrow$ separable). You can use this to prove 5.7.