We have seen 3 important categories of spaces

$$\begin{tikzcd} \text{Topological Sp.} \arrow[r, shift left, "3)"] & \text{Metric Spaces} \arrow[l, shift left, "2)"] \arrow[r, shift left, "4)"] & \text{Normed Spaces} \arrow[l, shift left, "1)"] \end{tikzcd}$$

1) <u>Norms induce metrics</u>

A norm on a vector space V over the reals $\mathbb{R}$ is a function

$$\|\cdot\| : V \longrightarrow \mathbb{R}_{\geq 0}$$

s.t.
a). $\|v\| = 0 \iff v = \underline{0}$
b). $\|\alpha v\| = |\alpha| \cdot \|v\|$ $\quad \forall v, w \in V$
c). $\|v+w\| \leq \|v\| + \|w\|$ $\quad \forall \alpha \in \mathbb{R}$

The <u>distance induced by $\|\cdot\|$</u> is given by

$$d : V \times V \longrightarrow \mathbb{R}_{\geq 0} \quad ; (x,y) \longmapsto d(x,y) := \|x-y\|$$

Let's prove it is indeed a distance

i) $d(x,y) = 0$ iff $x = y$

$d(x,y) = \|x-y\|$, and $\|x-y\| = 0$ iff $x - y = \underline{0}$ by a), hence $x = y$ ✓

ii) $d(x,y) = d(y,x)$

$d(x,y) = \|x-y\| = \|(-1)\, y - x\| = |-1|\, \|y-x\| = \|y-x\| = d(y,x)$ ✓

iii) $d(x,y) \leq d(x,z) + d(z,y)$

$d(x,y) = \|x-y\| = \|x - z + z - y\| \leq \|x-z\| + \|z-y\| =$
$= d(x,z) + d(z,y)$ ✓

2) A metric $d_X : X \times X \to \mathbb{R}_{\geq 0}$ induces a topology on X.

- d_X allows us to define balls
- $\mathcal{B} := \{ B(x_0, r) : x_0 \in X, r > 0 \}$
- Construct a collection of subsets of X using $\mathcal{B}$ as a 'basis'

$$\tau := \{ A \subseteq X \text{ s.t. } A = \bigcup_{i \in I} B_i \text{ , with } B_i \in \mathcal{B} \}$$

Then τ is a topology on X

~~proof~~
- $\emptyset, X \in \tau$ ✓
- $X_i \in \tau$ for all $i \in I$. WTS $\bigcup_i X_i \in \tau$

  $X_i = \bigcup_j B_j^i$ by assumption hence $\bigcup_i X_i = \bigcup_i \bigcup_j B_j^i \in \tau$ ✓
- $X_1, \dots, X_n \in \tau$. WTS $\bigcap_{i=1}^n \in \tau$

Let $x_0 \in \bigcap_{i=1}^n X_i$, it is enough if we find a ball $B \ni x_0$ with $B \subseteq \bigcap_{i=1}^n X_i$.

Since $x_0 \in X_i$ $\forall i = 1, \dots, n$, $\exists B_1, \dots, B_n$ s.t. $x_0 \in B_i$ and $B_i \subseteq X_i$ $\forall i = 1, \dots, n$.
Call r_i the radius of B_i, and x_i its radius.
Let

$$r_i' := r_i - d(x_0, x_i)$$

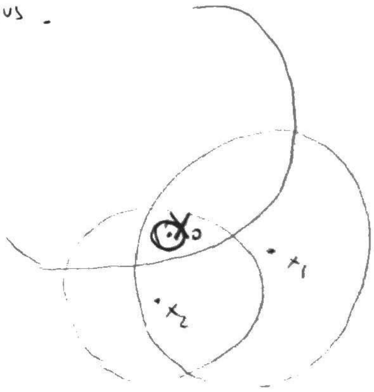
and let

$$r_0 < \min \{ r_1', \dots, r_n' \} .$$

Then

$$B(x_0, r_0) \subseteq \bigcap_{i=1}^n X_i$$

✓



Now let's see 3), 4).

First of all, a Remark

Remark 1 : X metric space is Hausdorff

Remark 2 : X metric space is 1st countable

Given $x_0 \in X$, build a local basis with balls centred in x_0, with rational radius.

...herefore we can ~~[illegible]~~ find counterexamples in non-Hausdorff or non first-countable spaces.

<u>Example 1</u> $(\mathbb{R}, \tau)$ with topology given by $\{\{I_x\}_{x\in\mathbb{R}}, \emptyset, \mathbb{R}\}$

where $I_x = (x, +\infty)$

is not Hausdorff

<u>Example 2</u> (<u>Line with two origins</u>)

- Take two copies of $\mathbb{R}$, call them $\mathbb{R}\times\{a\}$, $\mathbb{R}\times\{b\}$
- Glue them along every point $(x_0, a) \sim (x_0, b)$ $\forall x_0 \in \mathbb{R}\setminus\{0\}$

Hence we identify all the points, except the two zeros

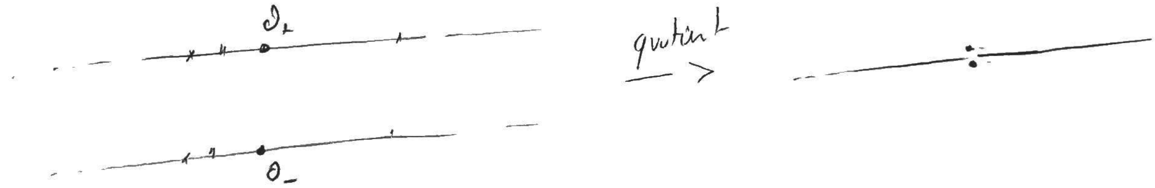
$\mathbb{R}\times\{+,-\}$ or

$$X = (\mathbb{R}\times\{a\}) \cup (\mathbb{R}\times\{b\}) \Big/ \sim$$

$(x_0, a) \sim (x_0, b)$ $\forall x \in \mathbb{R}\setminus\{0\}$

$[(0,a)] = \{(0,a)\}$ call it 0_+

$[(0,b)] = \{(0,b)\}$ call it 0_-



Non-Hausdorff: 0_+ and 0_- cannot be separated.

Intuitively, the points that we cannot separate are the ones where we get in troubles defining a metric

<u>Example 3</u>: $(\mathbb{R}, \tau)$ with $\tau = \emptyset \cup \{\mathbb{R}\setminus F : F \subseteq \mathbb{R} \text{ finite}\}$

in ex. 5.6 we saw it is not 1st countable, hence not metrizable

Let's see a more involved example

Example 4 : $[0,1]^{[0,1]}$, It is compact, but not sequentially compact

Build a sequence $\{f_n\}_n \subseteq X$

$$f_n : [0,1] \longrightarrow [0,1] : x \longmapsto \text{n-th digit of binary expression of } x.$$

then $\nexists f_{n_k}$ converging subsequence :

Suppose $\exists f_{n_k}$ converging $f_{n_k} \xrightarrow[k]{} f$ pointwise

Let $\bar{x} \in [0,1]$ s.t. n_1- th coord. of $\bar{x}$ is 0
n_2 - th coord of $\bar{x}$ is 1
n_3 - th coord of $\bar{x}$ is 0
0
1
0
⋮

Then $\{f_{n_k}(\bar{x})\}_k$ is the sequence $\{0,1,0,1,0 \dots\}$ which does not converge.

Hence f_{n_k} cannot converge to f

Then note the following fact

1st Countable + Compact $\Rightarrow$ Seq. Compact

And conclude that since X is compact and not seq. compact, follows that it cannot be 1st countable.

5) <u>Metric which is not induced by any norm</u>

Let's give one example for all.

Consider $\mathbb{R}$ with the <u>discrete metric</u> $\longrightarrow$ (induces discrete topology $B_{1/2}(x_0) = \{x_0\}$)

$$d(x,y) = \begin{cases} 1 & x \neq y \\ 0 & x = y \end{cases}$$

Such metric cannot be induced by a norm. We would have problems with property ii)

$x \neq 0$ $\quad \|x\| = d(x,0) = 1$. Let $\alpha > 1$. ~~[illegible]~~

$$\|\alpha \cdot x\| = |\alpha| \cdot \|x\| = |\alpha| \neq d(\alpha \cdot x, 0)$$

If you are curious about which conditions are required for a space to be metrizable / normable, we have two theorems characterizing them completely

- Nagata – Smirnov metrization theorem

  X top. space metrizable $\Longleftrightarrow$ regular, Hausdorff and has a countably locally finite basis

- Kolmogorov's normability criterion

  X top space normable $\Longleftrightarrow$ T1, admits bounded convex ngh of the origin.

Before stating the theorem, let us recall some definitions.
Let $I = [a,b] \subseteq \mathbb{R}$ be a compact interval in $\mathbb{R}$, and $\mathcal{C}(I) = \{f : I \to \mathbb{R} \text{ continuous}\}$ endowed with distance

$$d(f,g) := \sup_{x \in I} |f(x) - g(x)|.$$

You will see in Ex. 6.1 that $\mathcal{C}(I)$ is a complete topological space.

$\mathcal{Y} \subseteq \mathcal{C}(I)$ is said <u>uniformly bounded</u> if

$$\exists M \in \mathbb{R} \text{ s.t. } |f(x)| \leq M \quad \forall f \in \mathcal{Y},\ \forall x \in I$$

<u>Note</u> M is independent from f and x.

$\mathcal{Y} \subseteq C(I)$ is said <u>uniformly equicontinuous</u> (or just <u>equicontinuous</u>) if

$\forall \varepsilon > 0\ \exists \delta_\varepsilon$ s.t. whenever $|x-y| < \delta_\varepsilon$, we have $|f(x) - f(y)| < \varepsilon\ \ \forall f \in \mathcal{Y}$

<u>Note</u> δ may depend on ε, but not on x, y, f.

THEOREM (<u>Ascoli – Arzelà</u>, case real valued functions on compact interval)

Let $\mathcal{F} \subseteq \mathcal{C}(I)$.

$\mathcal{F}$ is relatively compact $\Longleftrightarrow$ $\mathcal{F}$ is uniformly bounded and equicontinuous

(COROLLARY: $\mathcal{F}$ compact $\Longleftrightarrow$ $\mathcal{F}$ closed, unif. bounded and equicontinuous
compare with Heine – Borel characterization of compactness)

PROOF: $\Longrightarrow$: $\mathcal{F}$ rel. compact, hence $\overline{\mathcal{F}}$ compact and hence $\overline{\mathcal{F}}$ tot. bounded.
Therefore $\mathcal{F}$ tot. bounded and hence $\mathcal{F}$ unif. bounded ✓

Let us fix $\varepsilon > 0$. By tot. boundedness $\exists f_1, \dots, f_N \in \mathcal{C}(I)$:

$$\mathcal{F} \subseteq B_{\varepsilon/3}(f_1) \cup \dots \cup B_{\varepsilon/3}(f_N) \qquad (\bullet)$$

Since each f_i is continuous, on a compact set, they are all unif. continuous (Heine – Cantor)

We fixed already ε. Then $\exists\, \delta$ s.t.

$$|f_i(x) - f_i(y)| < \varepsilon/3 \quad \text{whenever } |x-y| < \delta, \text{ and for all } i \in \{1, \dots, N\}$$

Given $f \in \mathcal{F}$, by $(\bullet)$, we know $\exists j$ s.t. $\|f - f_j\| < \frac{\varepsilon}{3}$

Now we can estimate, given $x, y \in I$ s.t. $|x-y| < \delta$

$$|f(x) - f(y)| \le |f(x) - f_j(x)| + |f_j(x) - f_j(y)| + |f_j(y) - f(y)| \le$$

$$\le \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon$$

Given the generality of $f \in \mathcal{F}$, we have $\mathcal{F}$ equicontinuous

<u>Hint</u>: ("$\varepsilon/3$-trick", may be useful in Ex. Sheet 6, Ascoli – Arzelà generalization)

$\Longleftarrow$: In order to show that $\mathcal{F}$ is relatively compact, we want to show that any sequence in $\mathcal{F}$ admits a converging subsequence

<u>strategy</u>: We will construct such a subsequence by building a sequence of nested subsequences, and then apply Diagonal argument.

- Fix an ordering of the rational points $\mathbb{Q} \cap [0,1]$ in $[0,1]$. Say $\{q_i\}_{i \in \mathbb{N}}$.

- Let $\{f_n\}_n$ be a generic sequence in $\mathcal{F}$.
  By uniform boundedness, the sequence of real numbers $\{f_n(q_1)\}_n \subseteq \mathbb{R}$ is bounded. Hence by Bolzano-Weierstrass

$$\exists \{f_i^{n_1}\}_i \subseteq \{f_n\}_n \text{ subsequence s.t. } \{f_i^{n_1}(q_1)\}_i \text{ converges.}$$

Now consider $\{f_i^{n_1}(q_2)\}_i \subseteq \mathbb{R}$, it is, again, bounded. Hence

$$\exists \{f_i^{n_2}\}_i \subseteq \{f_i^{n_1}\} \text{ s.t. } \{f_i^{n_2}(q_2)\}_i \subseteq \mathbb{R} \text{ converges}$$

note : $\{f_i^{n_2}(q_2)\}_i$ converges both in q_1 and q_2.

- Iterating this procedure, we get a sequence of nested subsequences

$$\{f_i^{n_1}\}_i \supseteq \{f_i^{n_2}\}_i \supseteq \{f_i^{n_3}\}_i \supseteq \dots \supseteq \{f_i^{n_K}\}_i \supseteq \dots$$

s.t. $\{f_i^{n_K}(q_j)\}_i$ converges for $q_j = q_1, q_2, \dots, q_K$

We are now ready to define the convergent subsequence. Let's write down more explicitly the sequences we have constructed

$$\begin{array}{ccccccc}
f_1^{n_1} & f_2^{n_1} & f_3^{n_1} & \cdots & f_K^{n_1} & \cdots \\
f_1^{n_2} & f_2^{n_2} & f_3^{n_2} & \cdots & f_K^{n_2} & \cdots \\
f_1^{n_3} & f_2^{n_3} & f_3^{n_3} & \cdots & f_K^{n_3} & \cdots \\
\vdots & & & & & \\
f_1^{n_K} & f_2^{n_K} & f_3^{n_K} & \cdots & f_K^{n_K} & \cdots
\end{array}$$

if we plug in q_1 or q_2 or q_3 in the seq. of functions, they all converge ← for example

Now we extract the diagonal, and define the subsequence $\{g_m\}$ of $\{f_n\}_n$

$$\boxed{g_m = f_m^{n_m}}$$

<u>Note</u> $\{g_m(q_i)\}_m$ converges for all q_i rational point in I, by construction.

Now let us show that such a subsequence $\{g_n\}_n$ is convergent.
We will do it by showing that it is Cauchy. Then the claim follows from the fact that $C(I)$ is complete.

Fix $\varepsilon > 0$. Let $\delta = \delta_\varepsilon$ be the delta of the def of equicontinuity.
By the compactness of $[0,1]$, we can cover it with finitely many intervals $I_1, \dots, I_J$ of width $< \delta$.

Let $t \in I$. We can estimate

$$|g_n(t) - g_m(t)| \le \overset{(1)}{|g_n(t) - g_n(q_i)|} + \overset{(2)}{|g_n(q_i) - g_m(q_i)|} + \overset{(3)}{|g_m(q_i) - g_m(t)|}$$

for t, q_i in the same I_j

(1) and (3) can be made $< \varepsilon/3$ for q_i close enough to t, by unif. equicont.

(2) can be made $< \varepsilon/3$ for big enough n, m, since $\{g_n(q_i)\}_n$ converges.

$$\le \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon$$

Hence $\{g_n\}_n$ is Cauchy. By completeness of $C(I)$, it converges $\square$

<u>Remark 1</u>: It could be shown that the convergence $g_n \to g_*$ is uniform.

That is because, in general $\{g_n\}_n$ seq. s.t. $g_n \longrightarrow g_*$ pointwise and $\{g_n\}_n$ unif. Cauchy, <u>then</u> $g_n \longrightarrow g_*$ uniformly.

<u>proof</u>: Fix $\varepsilon > 0$, Since $\{g_n\}$ is unif. Cauchy, $\exists N_\varepsilon$ s.t.

$$|g_m(x) - g_n(x)| < \varepsilon \quad \forall x \in I, \ \forall m, n > N$$

take limit for $m \to \infty$, get

$$|g_n(x) - g_*(x)| < \varepsilon \quad \forall x \in I, \ \forall n > N$$

hence $g_n \longrightarrow g_*$ uniformly $\square$

Remark 2 There are many ways of state Ascoli-Arzela thm, ..
many generalization.

It holds for $\mathcal{F} \subseteq \mathcal{C}(X, Y)$, where (X, d_X) compact metric space
(Y, d_Y) complete metric space.

We proved it for $X = [0,1]$
$Y = \mathbb{R}$

Try to 'get inspired' by this proof, and reproduce it in a more general setting.

Remark 3 A-A is very important.

·) characterizes compactness in an important class of spaces,
·) is central in many important proofs, and

·) gives us another instance of difference between fin. dim. and
∞-dim. vector space

Finite dim. and infinite dim. vector spaces behave very differently.

In ∞-dim. vector spaces, some typical properties of fin. dim. do not hold:

- $V \not\cong V^*$
- Linear mappings are not necessarily continuous
- Closed balls are not necessarily compact
- On a space V we can put two norms $\|\cdot\|^1$, $\|\cdot\|^2$ s.t. the $\|\cdot\|^1$-unit ball is unbounded respect to $\|\cdot\|^2$
- $\exists$ non-equivalent norms

Ex. 6.5, 6.6 are about exploring such difference between fin. and infinite linear spaces.
In particular, looking at norms

<u>Ex. 6.5</u> Prove that all the norms on a fin. dim. vector space are equivalent

- Fix $\|\cdot\|_\infty$ max norm.
  Show that a generic $\|\cdot\|$ is equivalent to $\|\cdot\|_\infty$
- Notice (prove it) $\|\cdot\| : (X, \|\cdot\|_\infty) \longrightarrow \mathbb{R}$ is continuous
- Use this fact to get $(\bullet)$

Ex. 6.6

(i) $\mathbb{R}^2$ is fin. dimensional!

The usual distance on $\mathbb{R}^2$ is d_{eucl}.

If we hope to find d' not equiv. to d, here we need to choose one which is not induced by a norm.

We have seen one $d'(x,y) = \begin{cases} 1 & x \neq y \\ 0 & x = y \end{cases}$

If d would be eq. to d', it would hold

$$d'(0, z_n) \leq C \, d(0, z_n) \qquad (\#)$$

You have d' and d, just need to construct a suitable seq z_n s.t. (#) cannot hold for all n

(ii) We have to fix an ∞-dim. vector space.

We know $C(I)$, $C^1(I)$

Fix two obvious norms, and try to repeat same argument as above.

Constructing a suitable seq. of functions.