Remark 3 Lecture 14

A <u>topological manifold</u> is a locally Euclidean, Hausdorff space.

It is important to underline that 'locally euclidean' and 'Hausdorff' are not redundant properties. Both are required to capture the notion of topological manifold

Example (locally Euclidean but not Hausdorff)

Consider two copies of the real line $\mathbb{R} \times \{-,+\}$,
Let $\sim$ be the equivalence relation identifying

$$(x,-) \sim (x,+) \qquad \forall\ x > 0$$

The eq. relation $\sim$ is, in terms of eq. classes:

$$[(x,-)] = [(x,+)] = \{(x,-),(x,+)\} \qquad \text{for } x > 0$$
$$[(x,-)] = \{(x,-)\} \qquad x \leq 0$$
$$[(x,+)] = \{(x,+)\} \qquad x \leq 0$$

Define $X := (\mathbb{R}\times\{-,+\})/\sim$

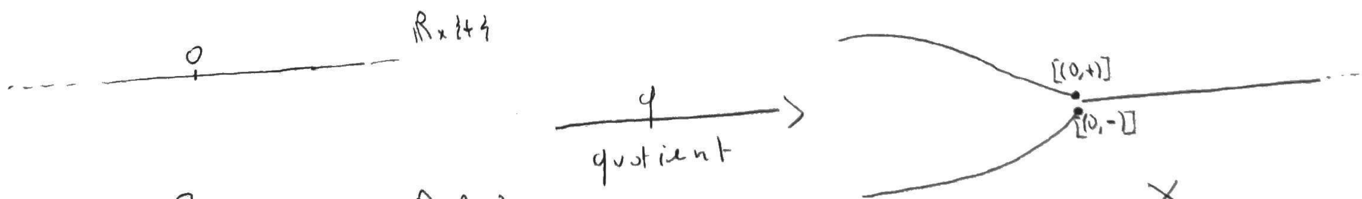
⚠ <u>Careful</u>: When working with quotient topology.
Don't confuse open of A with open set of $A/\sim$

Always keep in mind the difference.

X is defined as a quotient, and is endowed with quotient topology.
when we work with quotient topology, the golden rule is

$$\boxed{U \subseteq X \text{ open} \iff q^{-1}(U) \subseteq \mathbb{R}\times\{-,+\} \text{ open}}$$

How does X look like



Don't trust too much pictures. Trust this

<u>claim 1</u> : X is not Hausdorff

Let U be an open ngh <u>in X</u> of $[(0,-)]$
V be an open ngh <u>in X</u> of $[(0,+)]$

We want to show $U \cap V \neq \phi$.

By quotient topology def. $q^{-1}(U) \subseteq \mathbb{R} \times \{-,+\}$ open
$q^{-1}(V) \subseteq \mathbb{R} \times \{-,+\}$ open

Moreover, since $[(0,-)] \in U$ and $[(0,+)] \in V$, we have that
$(0,-) \in q^{-1}(U)$ and $(0,+) \in q^{-1}(V)$.

We know how open sets in $\mathbb{R} \times \{0,1\}$ look like
We know how open sets in $\mathbb{R} \times \{-,+\}$ ($\mathbb{R}$ usual topology, $\{-,+\}$ discrete topology. Hence open sets in $\mathbb{R} \times \{-,+\}$ are $\sigma \times D$ ~~open sets in $\mathbb{R}$~~ where σ open in $\mathbb{R}$ and D is $\{-\}$ or $\{+\}$ or $\{-,+\}$)

So $q^{-1}(U)$ contains an open set included in $\mathbb{R} \times \{-\}$, containing $(0,-)$
$q^{-1}(V)$ contains an open set included in $\mathbb{R} \times \{+\}$, containing $(0,+)$

In particular $q^{-1}(U)$ contains an open interval <u>in $\mathbb{R} \times \{-\}$</u>, containing $(0,-)$
say $q^{-1}(U) \supseteq \{(\lambda,-) : \lambda \in (-\varepsilon', \varepsilon')\} \overset{\text{notation}}{=:} ((-\varepsilon', \varepsilon'), -) =: \sigma_-$
Similarly $q^{-1}(V)$ contains an open interval in $\mathbb{R} \times \{+\}$, containing $(0,+)$
say $q^{-1}(V) \supseteq \{(\lambda,+) : \lambda \in (-\varepsilon'', \varepsilon'')\} \overset{\text{notation}}{=:} ((-\varepsilon'', \varepsilon''), +) =: \sigma_+$

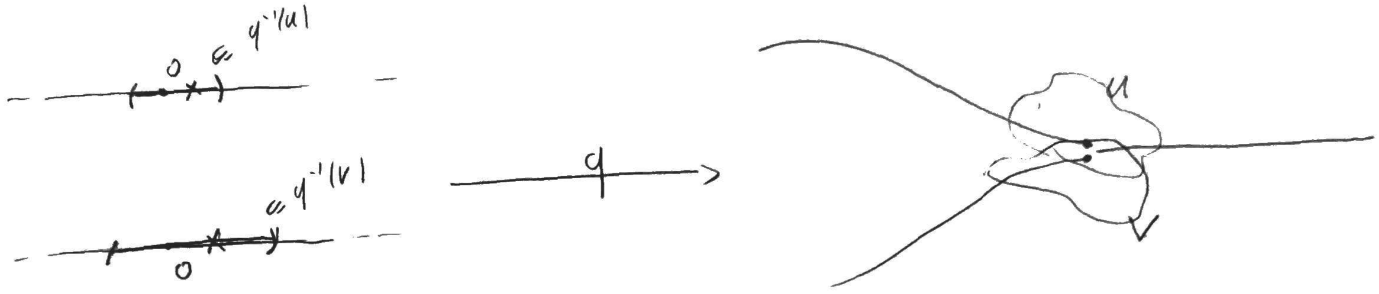
Let $\varepsilon := \min\{\varepsilon', \varepsilon''\}$.
we have that $(0 + \varepsilon/2, -) \in \sigma_-$ i.e. $(\varepsilon/2, -) \in \sigma_-$
$(0 + \varepsilon/2, +) \in \sigma_+$ $\quad (\varepsilon/2, +) \in \sigma_+$

But now notice that $q((\varepsilon/2,-)) = q((\varepsilon/2,+))$ because $\varepsilon/2 > 0$
So $(\varepsilon/2, -) \in \sigma_- \subseteq q^{-1}(U)$
$(\varepsilon/2, +) \in \sigma_+ \subseteq q^{-1}(V)$

hence $q((\varepsilon/2,-)) \in U$, $q((\varepsilon/2,+)) \in V$
but now notice that $q((\varepsilon/2,-)) = q((\varepsilon/2,+)) =: \bar{x}$ because $\varepsilon/2 > 0$

Hence $\bar{x} \in U \cap V$, and X is not Hausdorff.

In a picture



For ε small enough $(\varepsilon, -) \in q^{-1}(U)$
$(\varepsilon, +) \in q^{-1}(V)$

Mapping these points back in X, with q, they are mapped in the same point, which belongs to U and V resp.

$$\Rightarrow U \cap V \neq \emptyset$$

<u>Claim 2</u>: X is locally Euclidean

Let $x \in X$. Say $x = q(y)$, for some $y \in \mathbb{R} \times \{-\}$ w.l.o.g.

Fix $\varepsilon > 0$.

WTS: The restriction of the quotient map q on the open interval in $\mathbb{R} \times \{-\}$ containing y, say $\{(\lambda, -) ; \lambda \in (y-\varepsilon, y+\varepsilon)\} =: ((y-\varepsilon, y+\varepsilon), -)$ is a homeomorph. onto its image, which contains x, and hence provides the local homeomorph. of a ngh of x with

$$((y-\varepsilon, y+\varepsilon), -) \overset{\text{H}}{\simeq} (y-\varepsilon, y+\varepsilon) \subseteq \mathbb{R} \overset{\text{H}}{\simeq} \mathbb{R}.$$

Given the generality of x, it shows every $x \in X$ has a ngh homeom. to $\mathbb{R}$.

Call the interval in $\mathbb{R} \times \{-\}$, I.

$q_{|I}$ is trivially <u>surjective</u> onto its image $q_{/I}(I) \ni x$

<u>injective</u>: Yes. the only way the projection can fail to be injective is mapping two points of $\mathbb{R} \times \{-\}$ and $\mathbb{R} \times \{+\}$ in the same one in the quotient.

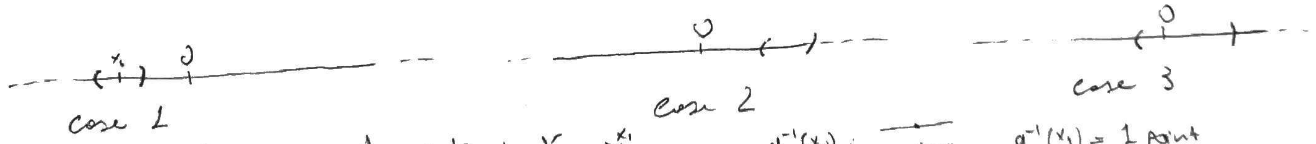
Since $I \subseteq \mathbb{R} \times \{-\}$, $q_{|I}$ is injective

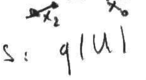
Hence $q_{|I} : I \longrightarrow q(I)$ is a bijection

Let's show now that <u>in this case</u> q is an open map.

Let U open in $\mathbb{R} \times \{-, +\}$. We can restrict to the case when U is an interval, say in $\mathbb{R} \times \{-\}$.
(The argument does not change if U is union of intervals, either in $\mathbb{R} \times \{-\}$ or $\mathbb{R} \times \{+\}$)

There are now 3 cases to distinguish



<u>Remark</u>: ~~the~~ Preimages of points in Y  $q^{-1}(x_0)$: $q^{-1}(x_1) = 1$ point
$q^{-1}(x_2) = 1$ point

<u>Case 1</u>: $U = ((x_0-\varepsilon, x_0+\varepsilon), -)$. WTS: $q(U)$ open.

In order to show if $q(U)$ open, we need to check if $q^{-1}(q(U))$ is open

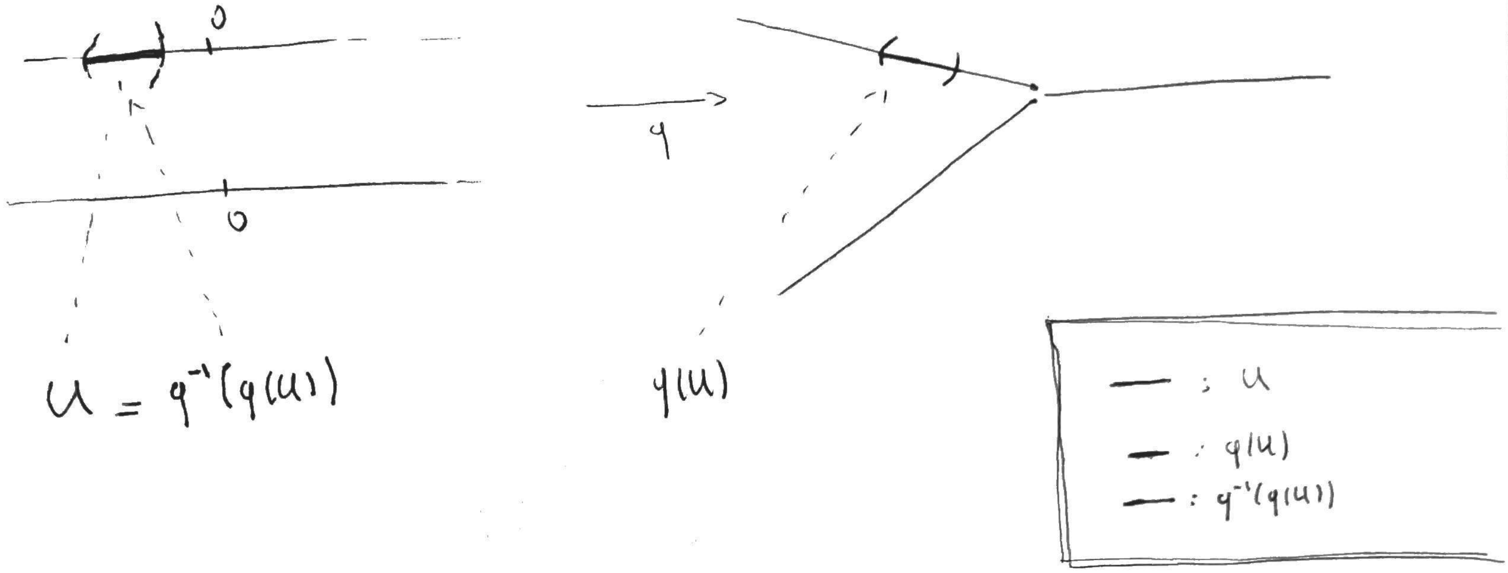
$$q(U) = q(((x_0-\varepsilon, x_0+\varepsilon), -)) = \{[(\lambda, -)] : \lambda \in (x_0-\varepsilon, x_0+\varepsilon)\} = \{\{(\lambda, -)\} : \lambda \in (x_0-\varepsilon, x_0+\varepsilon)\}$$

For $\lambda < 0$ we have $[(\lambda, -)] = \{(\lambda, -)\}$

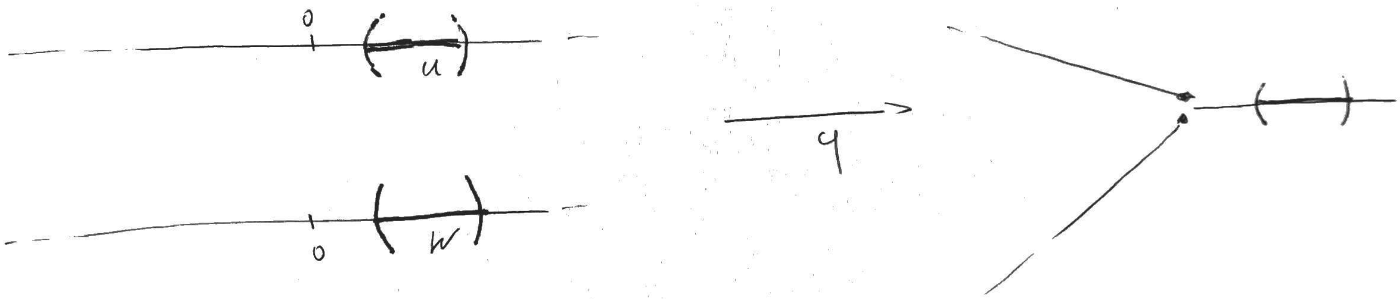
and hence $q^{-1}(q(U)) = U$, which is <u>open in $\mathbb{R} \times \{-, +\}$</u>

$\Rightarrow q(U)$ is open <u>in X</u>

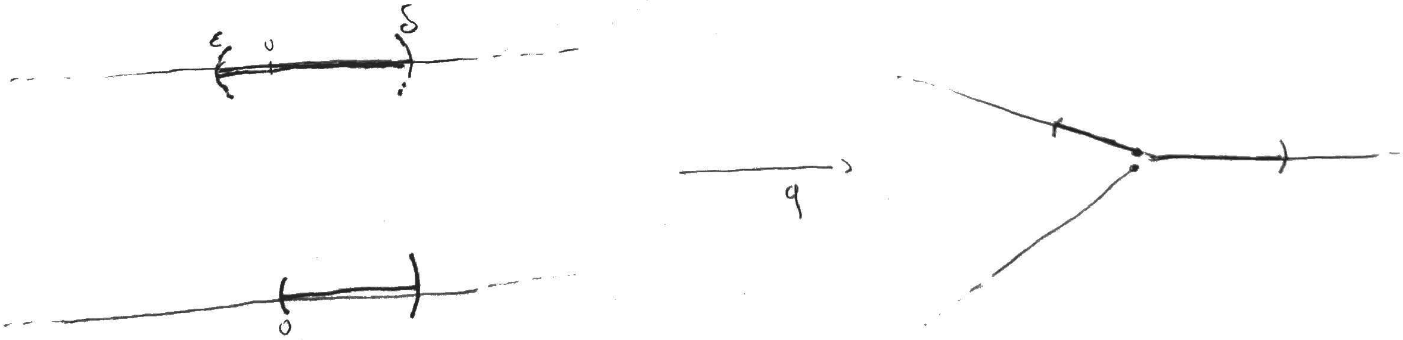
In a picture, what happens in Case 1 is



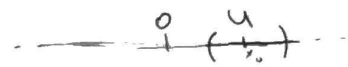
Case 2



Case 3



Case 2 : Let U be an open interval on the right of the zero, included in $\mathbb{R} \times \{-\}$. Say $U = ((x_0-\varepsilon, x_0+\varepsilon), -)$.



As before:

$$q(U) = \{[x] : x \in U\} = \{\{(x,-),(x,+)\} : x_0-\varepsilon < x < x_0+\varepsilon\}$$

Because remember that for $x > 0$, $(x,-) \sim (x,+)$

Therefore

$$q^{-1}(q(U)) = \underbrace{((x_0-\varepsilon, x_0+\varepsilon), -)}_{U} \cup \underbrace{((x_0-\varepsilon, x_0+\varepsilon), +)}_{=:W}$$

which is open in $\mathbb{R} \times \{-,+\}$. Hence $q(U)$ open in X ✓

Case 3 : U open interval in $\mathbb{R} \times \{-\}$, containing 0. Say $U = ((-\varepsilon, \delta), -)$.

$$q(U) = \{[x] : x \in U\} = \{\{(x,-)\} : x \leq 0\} \cup \{\{(x,-),(x,+)\} : 0 < x < \delta\}$$

Because when $x \leq 0$, every element is identified only with itself by $\sim$. And if $x > 0$, then we have $(x,-) \sim (x,+)$.

Therefore

$$q^{-1}(q(U)) = ((-\varepsilon, 0], -) \cup ((0,\delta), -) \cup ((0,\delta), +)$$
$$= ((-\varepsilon, \delta), -) \cup ((0,\delta), +)$$

which is again, open in $\mathbb{R} \times \{-,+\}$. Hence $q(U)$ open in X

The same argument applies when U is union of intervals.

To sum up : q is open ✓

Since I open in $\mathbb{R} \times \{-,+\}$, we have $q_{|I}$ is open, and hence $(q_{|I})^{-1}$ is continuous

Now we just are left to show $q_{|I}$ is continuous : Let $V \subseteq q(I)$ open in $q(I)$. Since $q(I)$ open in X (because q is an open map), the V is open in X. Then, by continuity of q, $q^{-1}(V)$ is open in $\mathbb{R} \times \{-,+\}$, and hence $q^{-1}(V) \cap I$ is open in I, as wanted ✓

To sum up

$$q_{|_I} : I \longrightarrow q(I) \ni x$$

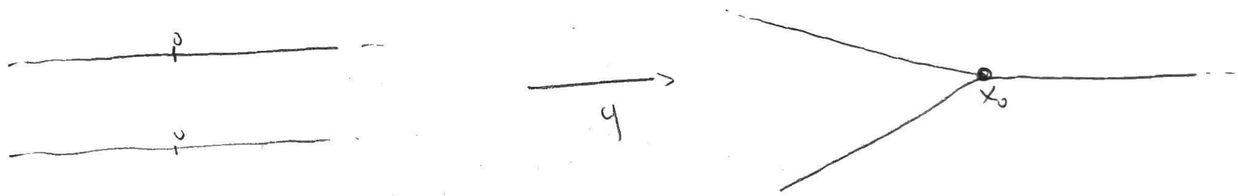
is a homeomorphism from $I \overset{H}{\cong} \mathbb{R}$ to a ngh $q(I) \ni x$.

Therefore X is locally euclidean $\square$

<u>Remark 1</u>: With just a small variation in $\sim$, we can make $X = (\mathbb{R} \times \{-,+\})/_\sim$ not loc. euclidean, but Hausdorff

$$(x,-) \sim (x,+) \quad \text{if } x \geq 0$$

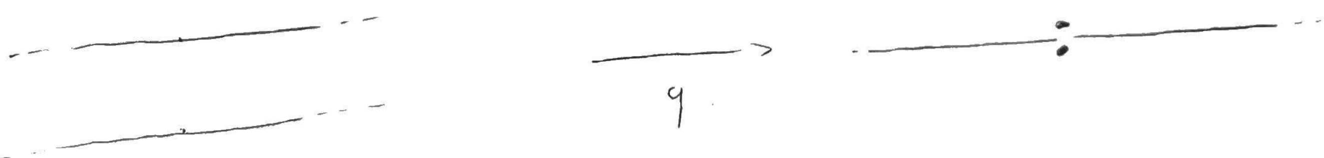
Then



Hausdorff, but locally every ngh of x_0 looks like "⅄" $\not\cong \mathbb{R}$

<u>Remark 2</u>: The exact same arguments hold for the example made last week: line with two origins

$$(x,-) \sim (x,+) \quad \forall x \neq 0$$



loc. euclidean, but not Hausdorff.

Ex 7.6 Show that the product space $S^1 \times S^1 =: X_1$ homeomorphic to the quotient space

$$X_2 = Q/_\sim$$

where $Q = [0,1]^2$ and $\sim$ eq. relation on Q identifying

$$(s,0) \sim (s,1) \quad \forall s \in [0,1]$$
$$(0,t) \sim (1,t) \quad \forall t \in [0,1]$$

PROOF, Consider the map

$$f : [0,1] \times [0,1] \longrightarrow S^1 \times S^1 \subseteq \mathbb{R}^4$$
$$(s,t) \longmapsto (\cos(2\pi s), \sin(2\pi s), \cos(2\pi t), \sin(2\pi t))$$

it obviously is a continuous map

Now, recall the universal property for topological quotients

X top. space, $\sim$ eq. rel. on X, $\pi : X \longrightarrow X/_\sim$ projection.

Let $f : X \longrightarrow Y$ be a continuous map s.t.

$\forall x_1, x_2 \in X : x_1 \sim x_2 \implies f(x_1) = f(x_2)$.

Then $\exists!$ continuous map $\tilde{f} : X/_\sim \longrightarrow Y$ s.t. $\tilde{f} \circ \pi = f$

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d] \arrow[dr, phantom, "\circlearrowright"] & Y \\ X/_\sim \arrow[ur, dashed, "\exists! \tilde{f}"'] & \end{tikzcd}$$

In our case we can see that

$$f(s,0) = (\cos(2\pi s), \sin(2\pi s), \cos 0, \sin 0) =$$
$$= (\cos(2\pi s), \sin(2\pi s), \cos 2\pi, \sin 2\pi) = f(s,1)$$

and similarly $f(0,t) = f(1,t)$ $\quad \forall s,t \in [0,1]$

therefore f satisfies the condition of the universal property, and hence it descends to a continuous fuction $\bar{f} : Q/_\sim \longrightarrow S^1 \times S^1$

$$\bar{f}([(s,t)]) = f(s,t)$$

WTS: $\bar{f}$ is a homeomorphism.

Recall the Homeomorphism Criterion (seen in class, lecture 9)

$f : X \to Y$ continuous bijection with X compact, Y Hausdorff

$\overset{\text{then}}{\implies}$ f is a homeomorphism.

In our case $Q/_\sim$ is compact being image under projection (i.e. continuous) of the compact space Q. $S^1 \times S^1$ is Hausdorff being subspace of Hausdorff space $\mathbb{R}^4$. Hence $\bar{f}$ homeomorph. □

Let X^* be a compact top. space, and $X \subseteq X^*$ a dense subspace.
We say X^* is a <u>compactification</u> of X. C(1)

We are interested in the situation where $X^* - X$ is just one point.

<u>Construction</u>: • Start with X given topological space.
• Define $X^* := X \cup \{\infty\}$, where ∞ is an element not in X
('∞' is just a symbol here, we could have used '$*$')

Hence we defined X^* by adding a new point to X

Notation ⚠
In order to distinguish complements in X and in X^*, denote
$A^{c*} = X^* - A \qquad A^c = X - A$
if $\infty \in A$. Then $A^c = A^{c*}$

• Define a topology on X^* as follows

$$A \in \tau^* \iff \begin{cases} A \subseteq X, \ A \in \tau \\ \text{or} \\ \infty \in A \text{ and } A^c \text{ is } \underline{\text{closed}} \text{ and compact in } X \end{cases} \quad \left(\begin{smallmatrix}\text{call this}\\ \text{family}\\ \mathcal{B}_\infty\end{smallmatrix}\right)$$

Hence $\tau^* = \tau \cup \mathcal{B}_\infty$, and $\tau \cap \mathcal{B}_\infty = \phi$

<u>Remark</u>, We are specifying A^c compact <u>and closed</u> because we are not assuming X Hausdorff. Hence compact sets are not automatically closed.

<u>CLAIM</u>: τ^* indeed defines a topology on X^*.

(i) $\phi \in \tau_*$ because $\phi \in \tau$
$X^* \in \tau_*$ because trivially $\infty \in X^*$ and $X^{*c^*} = \phi$ is closed and compact in X.

(ii) <u>Arbitrary union</u>.
Case 0: If $\{U_i\}_{i\in I}$ is a family of elements all in τ, then trivially $\bigcup_i U_i \in \tau$, because τ is already a top.
Case 1. $\{B_i\}_{i\in I}$ a family of elements in $\mathcal{B}_\infty$. WTS $\bigcup_{i\in I} B_i \in \mathcal{B}_\infty$

$$\Big(\bigcup_{i\in I} B_i\Big)^{c^*} = \bigcap_{i\in I} (X^* \setminus B_i) = \bigcap_{i\in I} \big(X \setminus (B_i \setminus \{\infty\})\big)$$

which is an arbitrary intersection of closed compact sets, hence again closed and compact

$$\Rightarrow \bigcup_{i\in I} B_i \in \mathcal{B}_\infty \ , \text{ i.e. } \in \tau_*$$

Case 2: If it is not the case that all the elements of the family are in $\mathcal{B}_\infty$

Consider $\{A_i\}_{i\in I}$. Then $\bigcup_{i\in I} A_i$ can be written as $A \cup B$ where B is an element in $\mathcal{B}_\infty$ and A is an element in τ

But then $(A \cup B)^{C^*} = A^{C^*} \cap B^{C^*}$. Since $\infty \in B$, then $\infty \notin B^{C^*}$, hence $\infty \notin A^{C^*} \cap B^{C^*} = (A \cup B)^{C^*}$. Therefore $(A \cup B)^{C^*} = (A \cup B)^C$ $= A^C \cap B^C$ which is intersection of a closed with a closed and compact, hence is again closed and compact.

$\Rightarrow A \cup B \in \tau_*$

To sum up $\{\vartheta_i\}_{i \in I}$ family of subsets in τ_*

$$\Rightarrow \bigcup_{i \in I} \vartheta_i \in \tau_*$$ and (ii) is verified

(iii) Case 0: If $\{U_1, \dots, U_n\}$ is a fin. collection of elements in τ, then trivially $\bigcap_{i=1}^n U_i \in \tau$, because τ is already a topology

Case 1: Let $\{B_1, \dots, B_n\} \subseteq \mathcal{B}_\infty$ all in $\mathcal{B}_\infty$. Then we have

$$\left(\bigcap_{i=1}^n B_i\right)^{C^*} = \bigcup_{i=1}^n B_i^{C^*} = \bigcup_{i=1}^n X \setminus (B_i \setminus \{\infty\})$$ which is finite union of closed, compact sets in X,

hence again closed and compact

$$\Rightarrow \bigcap_{i=1}^n B_i \in \mathcal{B}_\infty, \text{ i.e. in } \in \tau_*$$

Case 2: Let $\{\vartheta_i\}_{i \in I}$ finite family in τ_* not tot. included in $\mathcal{B}_\infty$ nor in τ. Then

$$\bigcap_{i \in I} \vartheta_i = U \cap B$$ by grouping together the elements in τ, and the ones in $\mathcal{B}_\infty$.

But then $U \cap B = U \cap (B \setminus \infty)$ because $\infty \notin U$
Recall $(B \setminus \infty)^C$ is closed in X. Hence $B \setminus \infty$ open in X

Therefore $U \cap (B \setminus \{\infty\})$ is intersection of two open in X, hence again open in X, i.e. $U \cap B \in \tau$

To sum up $\{\vartheta_i\}_{i \in I} \subseteq \tau_* \Rightarrow \bigcap_{i \in I} \vartheta_i \in \tau_*$
I finite

We therefore proved τ_* is a topology,
(X^*, τ_*) is a topological space.

The following result motivates why we called X^* a compactification of X.

PROPOSITION: X^* is compact, and X is open in X^*

PROOF: The open sets in τ are open in τ^*. Moreover, every open in τ^* intersects X in an open in τ

Follows that the topology τ is the topology induced by τ_* on X.
In particular, X open in τ_*, and X is dense in X^*.

Let us show X^* is compact.
Let $\{A_i\}_{i\in I}$ be an open covering of X^*. Then in particular $\exists\, i_0 \in I$ s.t. $\infty \in A_{i_0}$. Then $K := X \setminus A_{i_0}$ is closed and compact in X, and

$$\{A_i\}_{i\in I\setminus\{i_0\}} \text{ is an open covering of } K.$$

$\Rightarrow \exists$ finite subcover $\{A_i\}_{i\in J}$, J finite of $K = X \setminus A_{i_0}$.

But then $\{A_i\}_{i\in J} \cup \{A_{i_0}\}$ is a finite subcover of X^* from $\{A_i\}_{i\in I}$. Hence X^* is compact

COR/COMMENTS:
- $\{\infty\}$ is closed in X^*.
- if X is compact (X is always closed in X^* by PROP.), then $\{\infty\}$ is also open

that is because then $\{\infty\}^c = X$ compact, and closed.

On the other side, if $\{\infty\}$ is open, by def of τ_*, we have X is (closed and) compact

Summing up $\Longrightarrow$ X compact in X^* $\Longleftrightarrow$ $\{\infty\}$ is an isolated point in X^*

Now it makes sense to give the following def

Def. X non compact top. space. We call the space $X^* = X \cup \{\infty\}$ described so far, the <u>one-point compactification</u> of X

Examples

(1) The topology of the one-point compactification $\mathbb{R}^* = \mathbb{R} \cup \{\infty\}$ has basis

$$\mathcal{B}^* = \{(a,b) : a \in \mathbb{R}, b \in \mathbb{R}, a < b\} \cup \{[a,b]^{C^*} : a, b \in \mathbb{R}, a < b\}$$

(2) $\mathbb{N}$ natural numbers, with discrete topology.
A basis of ngh. around ∞ in $\mathbb{N}^*$ is given by the family

$$\{\{\infty\} \cup \{m \in \mathbb{N} : m \geq n\}\}_{m \in \mathbb{N}}$$

Comment: Compare this with definition of a sequence $a_n \xrightarrow[n]{} \infty$.

(3) $\mathbb{Q}^* = \mathbb{Q} \cup \{\infty\}$ is an example of a Frechet space (T1), not Hausdorff (not (T2)).

The space X^* in general does not need to be Hausdorff, but we know precisely when it is.

PROPOSITION: X^* is Hausdorff $\iff$ X is locally compact and Hausdorff

PROOF: $\Leftarrow$ Let X is loc. compact and Hausdorff.
In order to show X^* Hausdorff, is enough to show that given $x \in X$, we can find U ngh of x, V ngh of ∞, s.t. $U \cap V = \emptyset$.

Since X loc. compact, $\exists K$ compact ngh of x in X. Since X is open in X^*, K is a ngh of x in X^* as well.
Moreover X is Hausdorff. Hence K is closed. Therefore $V := K^{C^*}$ is a ngh of ∞, open. In particular $V \cap U = \emptyset$ and hence X^* is Hausdorff.

$\Rightarrow$: X^* Hausdorff. Then X is automatically Hausdorff.
Let us show that X is loc. compact. Let $x \in X$.
Since X^* is Hausdorff, then $\exists U, V$ ngh of x and ∞ resp. s.t. $U \cap V = \emptyset$. Therefore V^C is closed compact ngh of x.
Hence X loc. compact $\not{}$

<u>Example</u> : $(\mathbb{R}, \tau)$ when $\tau = \underbrace{\{ I_x = (-\infty, x) \}_{x \in \mathbb{R}}}_{\text{basis of } \tau} \cup \{ \emptyset, \mathbb{R} \}$

Then $(\mathbb{R}, \tau)$ is locally compact, but not compact.
(what are the compact sets in this topology? $(-\infty, M]$ for some M)

Follows from previous Prop. that X^* is not Hausdorff.
In fact, the only ngh of ∞ is the whole X^*.

Therefore, in order to obtain a "nice" (i.e. Hausdorff) compactification, we focus on the case X Hausdorff, locally compact, but not compact.

In this case, the one-point compactification X^* of X is called the <u>Alexandrov compactification</u>, from now on indicated with $\hat{X}$.

Alexandrov Compactification is <u>universal</u>, in the following sense

PROPOSITION : $\hat{X}$ Alexandrov compactif. of X loc. compact, Hausdorff, not compact.
Let $f : X \longrightarrow Y$ continuous map, with Y compact, Hausdorff, and such that $f(X)$ is the complement of a point in Y, and $f(X) \overset{H}{\simeq} X$
(i.e. Y has all the analogous properties of the Alexandrov compactification $\hat{X}$)

then $\Longrightarrow$ $\exists$ a unique homeomorph. $h : \hat{X} \longrightarrow Y$
s.t. $\forall x \in X \quad h(x) = f(x)$

In this sense <u>$\hat{X}$ is unique</u>.

In terms of diagrams :

$$\begin{tikzcd}
\hat{X} \arrow[dr, dashed, "\exists ! h \text{ homeomorph}"] & \\
X \arrow[u, phantom, sloped, "\subseteq"] \arrow[r, "f"'] \arrow[ur, phantom, "\circlearrowleft"] & Y
\end{tikzcd}$$

PROOF: Let us call p the only point in $Y \setminus f(X)$. Define $h : \hat{X} \longrightarrow Y$ as

$$h(x) = f(x) \qquad \text{for } x \in X$$
$$h(\infty) = p .$$

Since h is bijective, $\hat{X}$ is compact, Y is Hausdorff, in order to show that f is a homeomorph. it is enough to show that h is continuous. ('Proposition' in Ex. Sheet 7)

By construction, the restriction $h|_X : X \overset{\text{open}}{\subseteq} \hat{X} \longrightarrow f(X) \overset{\text{open}}{\subseteq} Y$ is continuous, we just need to check continuity on ∞

Let U an open ngh of p in Y. WTS $h^{-1}(U)$ is an open ngh of ∞ in $\hat{X}$.
U^c is closed, and Y is compact. Hence U^c is compact.
In particular $U^c \subseteq f(X)$ is compact in $f(X)$.
But then $f^{-1}(U^c)$ is compact in X. Since X is Hausdorff, follows that $f^{-1}(U^c)$ is closed in X. Hence

$$\left(f^{-1}(U^c)\right)^c = h^{-1}(U)$$

is open, and contains ∞. Hence h is continuous in ∞, and hence an homeomorph.

The uniqueness of h is trivial

$\not{f}$ $\square$

Now it finally make sense to refer to $\hat{X}$ as <u>the</u> Alexandrov compactification of X

<u>Example</u>: Recall stereographic projection

$\varphi : \mathbb{R} \xrightarrow{\cong} S^1 \setminus \{p\}$. $\qquad \hat{\mathbb{R}} = S^1$

In particular, $\hat{\mathbb{R}} \cong \widehat{S^1 \setminus \{p\}} = \hat{S}^1$ therefore the compactification "adds" the point p, which is the one ideally mapped by φ to ∞

Analogously $\hat{\mathbb{R}}^n \cong S^n$.