Exercise Class 20/04.

7.1. Consider the equivalence relation $\sim$ on $\mathbb{R}$:

$x \sim y$ iff $x = y$ or $|x| = |y|$, $|x| > 1$

Let $Y = \mathbb{R} \setminus \sim$. Show Y is not a H'dff space

first recall

$q : X \longrightarrow X/\sim$ is the projection sending

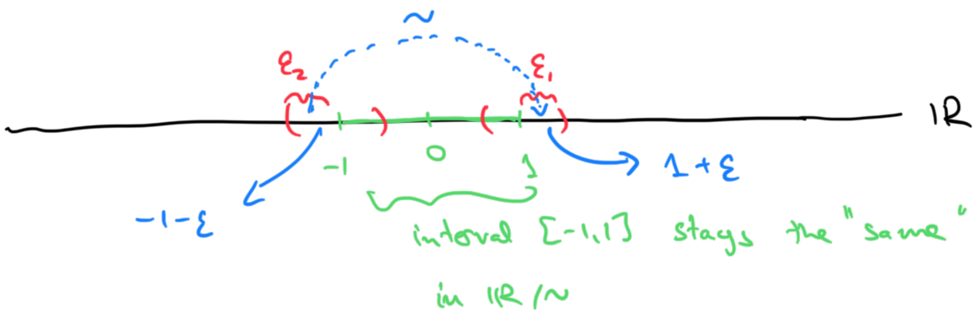
x to $[x]$

the topology on $X/\sim$ is defined by:

$U \subset X/\sim$ open $\Leftrightarrow$ $q^{-1}(U) \subset X$ open

Proof of 7.1

Picture:



by picture, the ε_2-nbhd of $-1 \neq 1$ (in Y) will intersect the ε_1-nbhd of 1 (in Y).

$q : \mathbb{R} \longrightarrow Y$ quotient map. $q(1) \neq q(-1)$.

Choose U, V open nbhd in Y of $q(1)$ and $q(-1)$ respectively

We show $U \cap V \neq \emptyset$.

$g^{-1}(U)$, $g^{-1}(V)$ are open in $\mathbb{R}$. In particular $\exists \varepsilon_1, \varepsilon_2$:
$(1-\varepsilon_1, 1+\varepsilon_1) \subseteq g^{-1}(U)$ and $(-1-\varepsilon_2, -1+\varepsilon_2) \subseteq g^{-1}(V)$
Let $\varepsilon < \min\{\varepsilon_1, \varepsilon_2\}$ we have $1+\varepsilon \in g^{-1}(U)$,
$-1-\varepsilon \in g^{-1}(V)$
But $g(1+\varepsilon) = g(-1-\varepsilon) \Rightarrow U \cap V \neq \emptyset$. $\square$.

7.2 (i) $\exists\, g : (-2,2) \to [-1,1]$ quotient map.

Soln We need to find $\sim$ equivalence relation such that

$$(-2,2)/\sim \ \underset{\text{homeo}}{\cong} [-1,1]$$

$$x \sim y \iff x, y \leq -1 \quad \text{or} \quad x, y \geq 1$$

check $\sim$ equivalence relation.

(ii) $\nexists\, p : [-2,2] \to (-1,1)$ quotient map.

Soln if p would be a quotient map, then p would a continuous surjection.
but $[-2,2]$ is compact $\Rightarrow p([-2,2])$ compact
$$p([-2,2]) = (-1,1) \text{ (not compact}$$
Contradiction.

7.3 X, Y top. spaces. $f : X \to Y$ continuous surjection
Y has the quotient topology:

U open in Y iff $f^{-1}(U)$ open in X.

(i) X compact then Y is compact.

True f is cts. surjective $\Rightarrow f(X)$ compact

(ii) X H'dff then Y is H'dff.

<u>False</u> by ex 7.1

(iii) X normal, then Y is H'dff.

$X = \mathbb{R}$ is normal.

<u>False</u> ex 7.1

(iv) if $|X| = \infty$, the $|Y| = \infty$.

<u>False</u> Consider the constant map:

$$f: \mathbb{R} \to \{0\}$$

($X \to X/\sim$ $\sim$: $x \sim y$ $\forall x, y$)

cts, surjective

$\{0\}$ has the quotient topology.

$X = \mathbb{N}$ $x \sim y$ iff $x - y$ even

$Y = \mathbb{N}/\sim$ $|Y| = 2$

$1 \sim 3 \sim 5 \sim \ldots$

$0 \sim 2 \sim 4 \sim \ldots$

$Y = \{[0], [1]\}$

(v) X connected, then Y is.

<u>True</u> Since f is continuous $\Rightarrow$ $f(X)$ is connected

$\overset{\|}{Y}$ $\to$ f surjective

(vi) X metric space, Y is.

<u>False</u> ex 7.1 $X = \mathbb{R}$ metric space

Y is not H'dff it can't be a metric space.

$X = \mathbb{R}$, $\quad Y$ line with origins.

7.4 X topological space and all connected components are open. $g : X \to Y$ quotient map of X.

Show all connected components of Y are open.

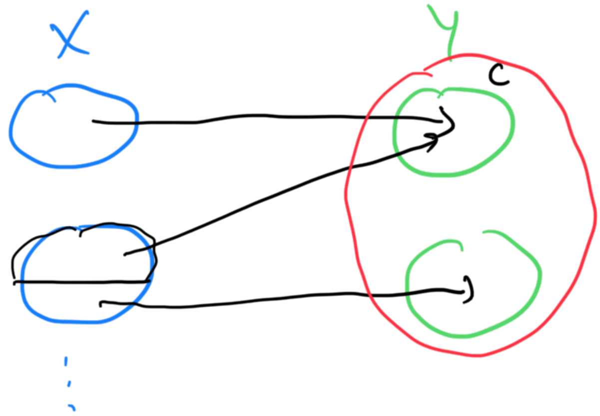
Interesting note $\quad X = \mathbb{Q} \subseteq \mathbb{R}$. Then all its connected components are singletons. Thus, none are open

Proof of 7.4 $\quad$ let C connected component of Y.

Enough to show $g^{-1}(C)$ is open in X.

We show $g^{-1}(C) = \bigcup_i K_i$, where K_i are connected comp of X.

Assume we have



but this shoul be connected since is connected.

let $K \subseteq X$ connected component of X,

$g(K) \cap C \neq \emptyset$. We want to show $g(K) \subseteq C$.

$g(K)$ connected (g is cts)

$\Rightarrow g(K) \cup C$ is connected

C is a connected component then $g(K) \cup C = C$.

Hence $g(K) \subseteq C$

$\Rightarrow g^{-1}(C) = \bigcup_i K_i$, K_i connected components of

$\underset{K_i \text{ open.}}{\Rightarrow} g^{-1}(C)$ is open $\underset{g \text{ quotient map.}}{\Rightarrow}$ C is open in Y

7.9. X topological space. $\Delta = \{(x,y) \in X \times X : x = y\}$ diagonal of $X \times X$.

(i) X H'dff iff Δ is closed in $X \times X$

<u>Proof</u>

"$\Rightarrow$" Assume X is H'dff.

We show that $Y = X \times X \setminus \Delta$ is open.

Let $(x,y) \in Y$. then $\exists\, U, V$ open in X (X is H'dff.)

$x \in U$, $y \in V$, $U \cap V = \emptyset$

$\Rightarrow U \times V$ open in $X \times X$.

if $\exists\, z$ s.t. $(z,z) \in U \times V$ then $z \in U \cap V$. ↯

Thus $U \times V \subseteq Y$.

$\Rightarrow Y$ is open $\Rightarrow \Delta$ closed in $X \times X$.

"$\Leftarrow$" Assume Δ closed in $X \times X$ i.e.

$Y = X \times X \setminus \Delta$ is open.

Take $x \neq y \in X \Rightarrow (x,y) \in Y$.

$\exists\, O$ open subset of Y , $(x,y) \in O$.

We can write $O = \bigcup_i U_i \times V_i$, U_i, V_i open in

then $\exists\, i_0$ s.t. $(x,y) \in U_{i_0} \times V_{i_0} \subseteq O \subseteq Y$

Since $U_{i_0} \times V_{i_0} \subseteq Y$ then $U_{i_0} \cap V_{i_0} = \emptyset$

Hence X is H'dff ($x \in U_{i_0}$, $y \in V_{i_0}$)

if $z \in U_{i_0} \cap V_{i_0} \Rightarrow (z,z) \in U_{i_0} \times V_{i_0}$ ↯ because $U_{i_0} \times V_{i_0} \subseteq Y$

(ii) $\sim$ equivalence relation $R = \{(x,y) \in X\times X,\ x\sim y\}$

Suppose $g : X \longrightarrow X/\sim$ ~~open.~~ Then

closed "$\Leftarrow$"

$X/\sim$ H'dff iff R closed in $X\times X$.

"$\Rightarrow$" Assume $X/\sim$ H'dff. We show $X\times X\setminus R$ is op

let $(x,y) \in X\times X \setminus R$. Then $g(x) \neq g(y)$

by def of R.

Then $\exists\, U_x, U_y$ open disjoint in $X/\sim$

such that $g(x) \in U_x$, $g(y) \in U_y$.

$g^{-1}(U_x)$, $g^{-1}(U_y)$ are open in X

then claim $g^{-1}(U_x) \times g^{-1}(U_y) \subseteq X\times X \setminus R$

Indeed if $(u,v) \in \left(g^{-1}(U_x) \times g^{-1}(U_y)\right) \cap R$

then $u \sim v \Rightarrow \underset{\substack{\cap \\ U_x}}{g(u)} = \underset{\substack{\cap \\ U_y}}{g(v)}$ $\lightning$ contradiction.

Thus, $X\times X \setminus R$ open $\Rightarrow$ R is closed in $X\times X$.

"$\Leftarrow$" Assume that R is closed.

We show $X/\sim$ is H'dff

We will not repeat the argument in (i) since we show that $X/\sim$ is H'dff. Indeed, from this it does follow that X is also H'dff.

by ex 2.4 $g\times g : X\times X \longrightarrow (X/\sim)\times(X/\sim)$ is open (since g is open).

Since R is closed, $X\times X\setminus R$ is open then

$((X\times X)\setminus R)$ open set in $(X/\sim)\times(X/\sim)$

$g \times g\,((X \times X) \setminus R) = \dots$

$\parallel$ Claim

$((X/\sim) \times (X/\sim)) \setminus \Delta_\sim$

$\Delta_\sim = \{ (x,y) \in (X/\sim) \times (X/\sim) : x = y \}$

then by (i) we conclude (use (i) $X := X/\sim$)

Proof Claim

- $(u,v) \in X \times X \setminus R$

$\Rightarrow g \times g(u,v) = (g(u), g(v))$

$g(u) \neq g(v)$ by def of R

$\Rightarrow g \times g\,(u,v) \in ((X/\sim) \times (X/\sim)) \setminus \Delta_\sim$

$\Rightarrow g \times g\,(X \times X \setminus R) \subseteq ((X/\sim) \times (X/\sim)) \setminus \Delta_\sim$

- let $(u,v) \in ((X/\sim) \times (X/\sim)) \setminus \Delta_\sim$

$u \neq v$, $\exists\; g(x) = u,\; g(y) = v$

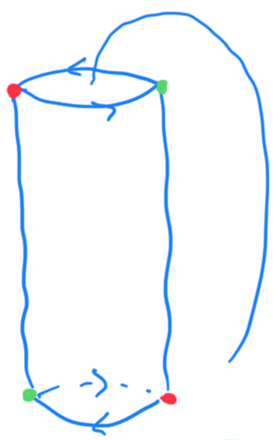
$g : X \to X/\sim$ surjective.

$u \neq v \overset{\text{by def}}{\Rightarrow} x \not\sim y \Rightarrow (x,y) \in X \times X \setminus R$

$\Rightarrow ((X/\sim) \times (X/\sim)) \setminus \Delta_\sim \subseteq g \times g\,(X \times X \setminus R)$

$\square$.

Klein bottle.



$(s,0) \sim (s,1) \quad \forall s \in [0,1]$

$(0,t) \sim (1, 1-t) \quad \forall t \in [0,1]$

in $\mathbb{R}^4$
does not cross in $\mathbb{R}^4$.