A frequent mistake on Problem Set 7:

If $q: X \to Y$ is a quotient map, and $f: X \to Z$ continuous and induces a bijection $\tilde{f}: Y \to Z$

$$\begin{tikzcd} X \arrow[d, "q"'] \arrow[dr, "f"] \arrow[d, phantom, shift left=4ex, "="] & \\ Y \arrow[r, "\tilde{f}"] & Z \end{tikzcd}$$

Then $\tilde{f}$ is continuous and bijective but <u>NOT</u> necessarily a homeomorphism / open.

<u>Ex:</u> $X = Y = (\mathbb{R}, \tau_{std})$, $Z = (\mathbb{R}, \overset{\{\emptyset, \mathbb{R}\}}{\tau_{triv}})$, $f = q = \tilde{f} = id_{\mathbb{R}}$.

$$\begin{tikzcd} (\mathbb{R}, \tau_{std}) \arrow[d, "id"] \arrow[dr, "id"] \arrow[dr, phantom, shift right=3ex, "="] & \\ (\mathbb{R}, \tau_{std}) \arrow[r, "id"] & (\mathbb{R}, \tau_{triv}) \end{tikzcd}$$

Not a homeom!

$(\mathbb{R}, \tau_{triv})$ is <u>not</u> <u>Hausdorff</u>!

8.2. Functions on contractible spaces ✍. Prove the following statements.

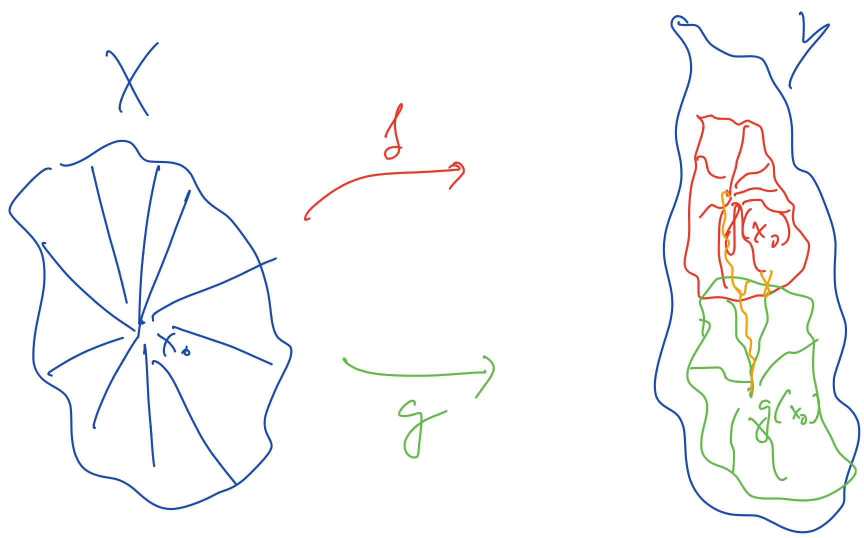
(i) Let X be a path-connected topological space. Show X is contractible if and only if for any path-connected topological space Y and any pair of functions $f, g\colon X \to Y$, we have that f and g are homotopic.

(ii) Show that a topological space X is contractible if and only if for any topological space Y and any pair of continuous function $f, g\colon Y \to X$, we have that f and g are homotopic.

Proof: (i) "⇒" Let X be contractible, let $f, g : X \to Y$ be cont. Denote by $H: X \times [0,1] \to X$ a homotopy between $id|_X = H(\cdot, 0)$ and the constant map $c_{x_0} = H(\cdot, 1)$ for some $x_0 \in X$. Set $y_1 = f(x_0)$, $y_2 := g(x_0)$, and let $\gamma: [0,1] \to X$ be a path from $\gamma(0) = y_1$ to $\gamma(1) = y_2$. Now, define

$F: X \times [0,1] \to Y$ by

$$F(x,s) := \begin{cases} f(H(x, 3s)) & \text{if } s \in [0, 1/3] \\ \gamma(3s-1) & \text{if } s \in [1/3, 2/3] \\ g(H(x, 3-3s)) & \text{if } s \in [2/3, 1] \end{cases}$$



Verify F is continuous. Moreover,

$$F(x,0) = f(H(x,0)) = f(x)$$

$$F(x,1) = g(H(x,\underbrace{3\cdot 1-3}_{=0})) = g(x)$$

$\Rightarrow$ F is a homotopy between f and g.

"$\Leftarrow$" Suppose every $f,g : X \to Y$ are homotopic.

Set $Y := X$ and consider $f := id_X : X \to X$

and $g := c_{x_0} : X \to X$, $x \mapsto x_0$ for some $x_0 \in X$.

$\Rightarrow$ $id_X \simeq c_{x_0}$ $\Rightarrow$ X is contractible

$\square$

(ii) "$\Rightarrow$" Suppose X is contractible and let $f,g : Y \to X$ be cont. Let $H : X \times [0,1] \to X$ be a homotopy between c_{x_0} and id_X. Define

$$F : Y \times [0,1] \to X$$

$$F(y,s) = \begin{cases} H(f(y), 2s) & \text{if } s \in [0,\frac{1}{2}], \\ H(g(y), 2-2s) & \text{if } s \in [\frac{1}{2},1] \end{cases}.$$

As before F is continuous and

$$F(y,0) = H(f(y),0) = f(y)$$

$$F(y,1) = H(g(y),0) = g(y) \quad \forall y \in Y$$

"$\Leftarrow$": Consider $X = Y$, $f = id_X$, $g = c_{x_0}$.

$\Rightarrow \exists$ a homotopy between id_X and c_{x_0}.

$\Rightarrow$ X is contractible $\square$

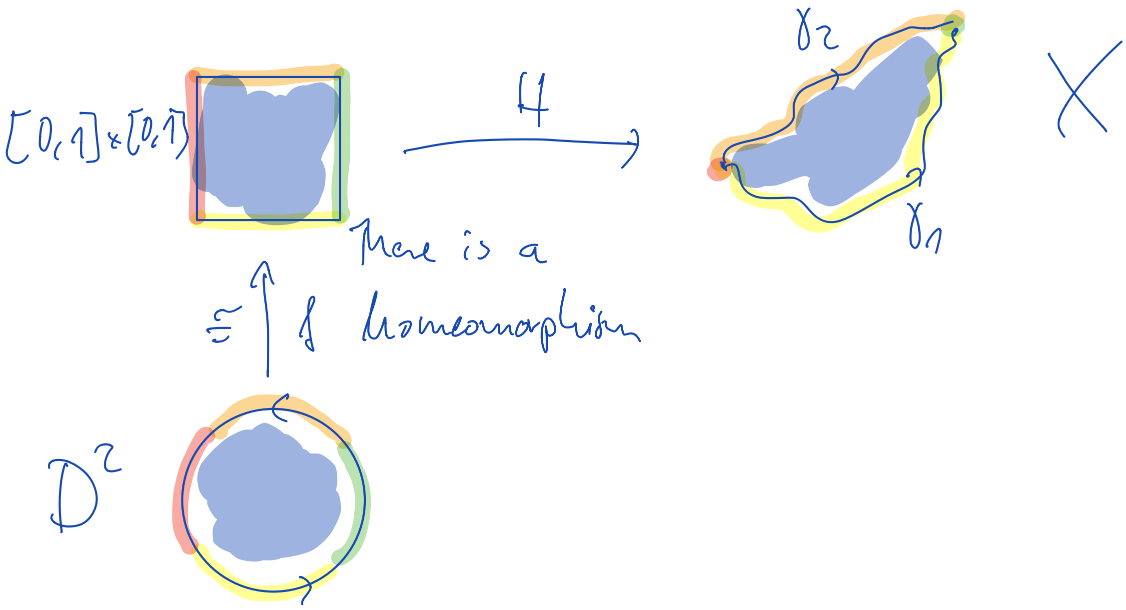
8.3. Homotopic paths. Let X be a topological space, and let γ_1, γ_2 be paths in X with the same endpoints (i.e. $\gamma_1(0) = \gamma_2(0)$ and $\gamma_1(1) = \gamma_2(1)$). Show that γ_1 and γ_2 are homotopic if and only if there is a continuous map $F\colon D^2 \to X$ such that $F|_{\partial D^2}\colon S^1 \to X$ is a reparametrization of $\gamma_1 * \gamma_2^{-1}$.

"$\Rightarrow$": Let $\gamma_1, \gamma_2 : [0,1] \to X$ be homotopic.

Then there is a homotopy

$H: [0,1] \times [0,1] \to X$

between $\gamma_1(t) = H(t,0)$ and $\gamma_2(t) = H(t,1)$:



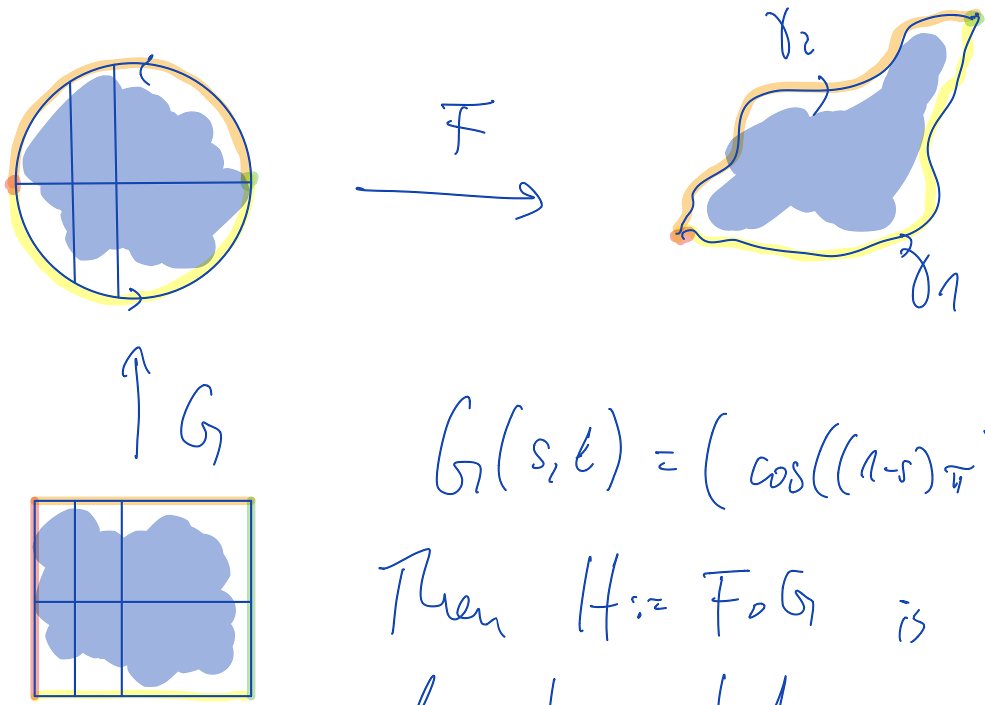
Set $F := H \circ f : D^2 \to X$. Then $F|_{\partial D^2} : S^1 \to X$ is a reparam. of $\gamma_1 * \gamma_2^{-1}$.

"$\Leftarrow$": Let $F: D^2 \to X$ be continuous, s.t. $F|_{\partial D^2}$ is a reparametrization of $\gamma_1 * \gamma_2^{-1}$.

Up to precomposition with another cont. map we may wlog assume that:

$\gamma_1(s) = F\left(\cos((1+s)\pi), \sin((1+s)\pi)\right), s \in [0,1]$

$\gamma_2(s) = F\left(\cos((1-s)\pi), \sin((1-s)\pi)\right), s \in [0,1].$



$G(s,t) = \left(\cos((1-s)\pi), (2t-1)\sin((1-s)\pi)\right)$

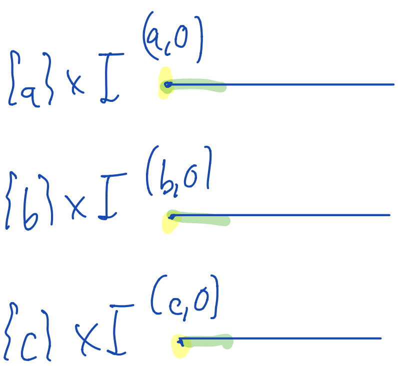
Then $H := F \circ G$ is the required homotopy between γ_1 and γ_2.

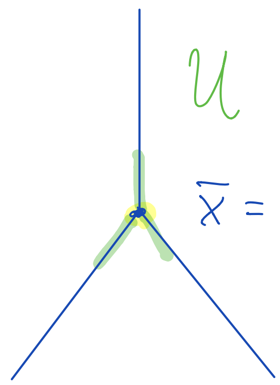
$\square$

8.6. Cone that is not a topological manifold ✎. Find an example of a topological manifold X such that the cone $C(X) := (X \times [0,1])/(X \times \{0\})$ over X is not a topological manifold around $\bar{x} = [X \times \{0\}] \in C(X)$, i.e. the point $\bar{x}$ does not admit any neighborhood $U \subseteq C(X)$ of $\bar{x}$ homeomorphic to $\mathbb{R}^n$ for some $n \in \mathbb{N}$.

Example: Take $X := \{a, b, c\}$ with discrete topology (X is a 0-dim. top. manifold).

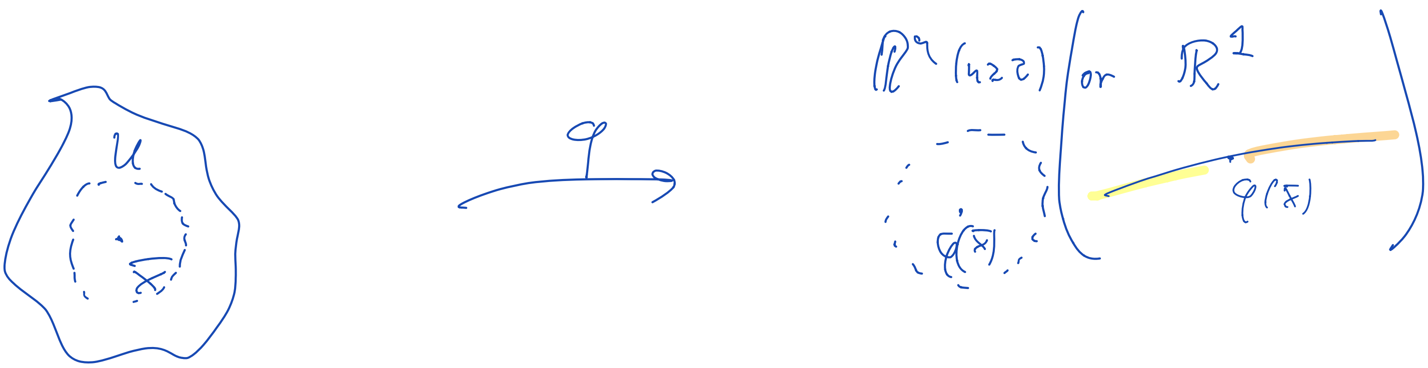
$X \times I$





$\bar{x} = [a,0] = [b,0] = [c,0]$

Suppose $\bar{x}$ admits an open nbhd. U that is homeomorphic to $\mathbb{R}^n$ via $\varphi : U \xrightarrow{\cong} \mathbb{R}^n$. In particular, $U \setminus \{\bar{x}\}$ has at most two connected components



But $U \setminus \{\bar{x}\}$ has $\underline{\underline{3}}$ conn. components. ↯

8.7. Every cone is contractible. Given a topological space X, let $C(X)$ denote the cone over X, i.e. $C(X) := (X \times [0,1])/(X \times \{0\})$. Show that

(i) $C(X)$ is path-connected;

(ii) $C(X)$ is contractible.

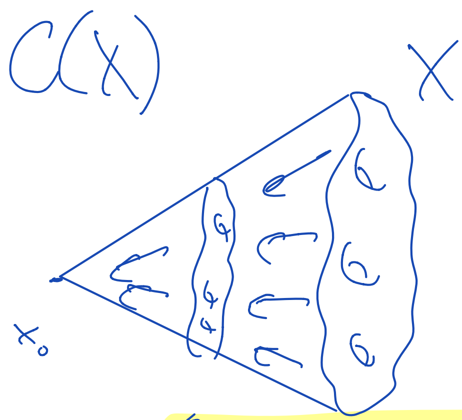
Proof: By 8.1 it suffices to show that $C(X)$ is contractible. Set $x_0 := q(x,0) \in C(X)$ where $q: X \times I \longrightarrow C(X) = (X \times I)/X \times \{0\}$ is the quotient map. We are looking for a homotopy between the constant map c_{x_0} and $id_{C(X)}$. Define

$$H: (X \times I) \times I \longrightarrow X \times I$$

by $H((x,t),s) := (x, (1-s)t)$

This descends to a map $\tilde{H}: C(X) \times I \to C(X)$:

$$\begin{tikzcd} (X \times I) \times I \arrow[r, "H"] \arrow[d, "q \times id"'] & X \times I \arrow[d, "q"] \\ C(X) \times I \arrow[r, dashed, "\tilde{H}"] & C(X) \end{tikzcd}$$



given by $\tilde{H}([x,s],t) = [x,(1-s)t]$

"$\tilde{H}$ retracts to the tip of the cone"

One can check that $q \times id$ is a quotient map

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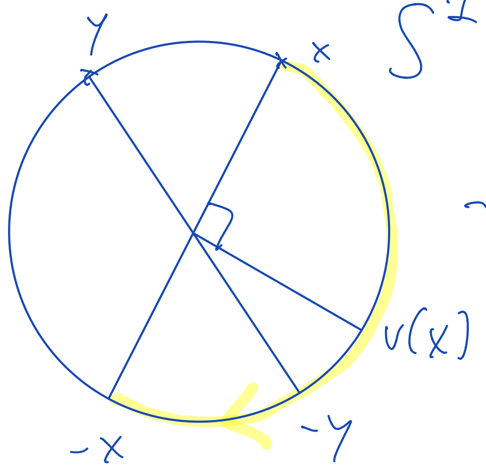
such that H is indeed continuous (by defn of the quotient map property). In fact, we prove in the solution:

<u>Lemma:</u> If $p: X \to Y$ is a quotient map then $(p \times id): X \times I \to Y \times I$ is a quotient map!

It is, in general, not true that the product of two quotient maps is again a quotient map!

8.8. Homotopic maps on the sphere ⚙. Show that, if $n \in \mathbb{N}$ is odd, the antipodal map $-\mathrm{Id}_{S^n} : S^n \to S^n$ on the sphere is homotopic to the identity map Id_{S^n}.

Proof: Sketch $n = 1$:



Note that $-\mathrm{Id}_{S^1}$ is simply a rotation by π!

$\rightsquigarrow$ We can just rotate back to Id_{S^1}.

Same idea works for any $n = 2k-1$:

Let $x = (x_1, y_1, x_2, y_2, \dots, x_k, y_k) \in S^n \subset \mathbb{R}^{2k}$

Consider the map

$$v(x_1, y_1, \dots, x_k, y_k) = (y_1, -x_1, y_2, -x_2, \dots, y_k, -x_k)$$

Note that $v(x) \in S^n$ and $\langle x, v(x) \rangle = 0$.

Define the homotopy $H : S^n \times [0,1] \to S^n$ „via rotation in every (x_i, y_i)-plane":

$$H(x,t) = \cos(\pi t) \cdot x + \sin(\pi t) \cdot v(x)$$

$\square$

Deformation retract to a point VS Contractible

Defn: X is <u>contractible</u> if the identity map $id_X : X \to X$ is homotopic to the constant map $c_{x_0} : X \to X$, $x \mapsto x_0$ for some $x_0 \in X$, i.e. there is a continuous map $H : X \times I \to X$ such that

(i) $H(x,0) = x \quad \forall x \in X$, and

(ii) $H(x,1) = x_0 \quad \forall x \in X$.

Here the homotopy H may <u>move</u> the point x_0, i.e. $H(x_0,t) = x_0$ DOES <u>NOT</u> necessarily hold $\forall t \in I$.

Defn: X <u>deformation retracts to a point $x_0 \in X$</u> if there is a continuous map $H : X \times I \to X$ such that

(i) $H(x,0) = x \quad \forall x \in X$,

(ii) $H(x,1) = x_0 \quad \forall x \in X$,

<u>AND</u> (iii) $H(x_0,t) = x_0 \quad \forall t \in I$.

These are DIFFERENT NOTIONS, and there are spaces that are contractible that do NOT def. retract to a point (8.9 & 8.10)

Here are the corresponding exercises in [Hatcher]:

6. (a) Let X be the subspace of $\mathbb{R}^2$ consisting of the horizontal segment $[0,1]\times\{0\}$ together with all the vertical segments $\{r\}\times[0,1-r]$ for r a rational number in $[0,1]$. Show that X deformation retracts to any point in the segment $[0,1]\times\{0\}$, but not to any other point. [See the preceding problem.]

(b) Let Y be the subspace of $\mathbb{R}^2$ that is the union of an infinite number of copies of X arranged as in the figure below. Show that Y is contractible but does not deformation retract onto any point.

$Y =$