Exercise class May 4th

lunedì 4 maggio 2020 09:56

Plan:

- Common mistakes / confusions on PS8
- Reminders on covering maps
- PS 9 : 9.3, 9.4
- 9.7 + discrete subsets
- Challenge problem

Common mistakes / confusions on PS8

(1) Difference between free homotopy and $\underbrace{\text{based homotopy}}_{\text{homotopy of paths}}$

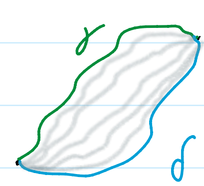
- $f, g : X \to Y$ are homotopic if $\exists\ H : X \times \underset{[0,1]}{I} \to Y$ s.t.

$H(-,0) = f$; $H(-,1) = g$.

$\leadsto$ two paths $\gamma, \delta : I \to X$ are (freely) homotopic if

$\exists\ H : I \times I \to X$ s.t. $H(-,0) = \gamma$ $H(-,1) = \delta$.

- $\gamma, \delta : I \to X$ s.t. $\gamma(0) = \delta(0)$ $\gamma(1) = \delta(1)$.

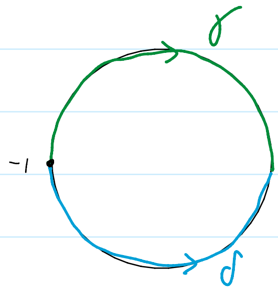


They are homotopic as paths / homotopic relative to their endpoints

if $\exists\ H : I \times I \to X$ s.t.
1) $H(-,0) = \gamma$; $H(-,1) = \delta$
2) $H(0,-) = \gamma(0) = \delta(0)$
$H(1,-) = \gamma(1) = \delta(1)$

Ex

EX:



Then γ is freely homotopic to δ (both homotopic to $c_{-1} : I \to S^1$)

<u>But</u> not homotopic rel. to their endpoints.

Indeed $\gamma \simeq \delta \Rightarrow [\gamma * \delta^{-1}] = id \in \pi_1(S^1)$

However $\gamma * \delta^{-1}$ is a generator of $\pi_1(S^1) \cong \mathbb{Z}$.

② Homotopy of constant maps

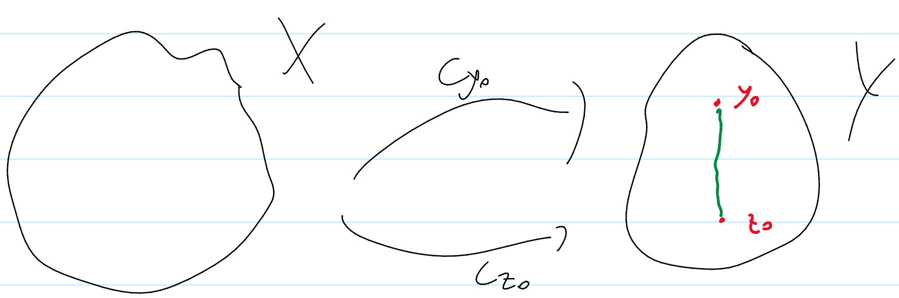
<u>Suggestion</u>: When talking about a constant map $X \to Y : x \mapsto y_0$.

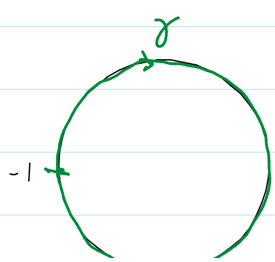
Write $\boxed{c_{y_0}}$, <u>not</u> y_0.

I have seen the following mistakes, $f, g : X \to Y$

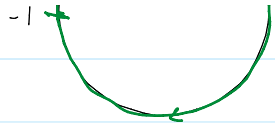
$$f(x) \simeq g(x) \quad \forall x \quad \overset{!!!}{\cancel{\Rightarrow}} \quad f \simeq g$$

In a path-connected space Y, all constant maps $X \to Y$ are homotopic.





$\gamma \not\simeq c_{-1}$, but S^1 is path-connected, so $\forall$ t



path-connected, so $\forall\, t$

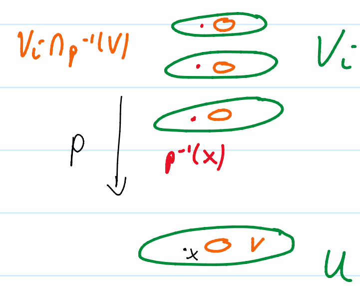
$c_{\gamma(t)} \simeq c_{-1}$

Reminders on covering maps

$p : \tilde{X} \to X$ continuous.

Def: $U \subseteq X$ is <u>evenly covered</u> if $p^{-1}(U) = \coprod_{i \in I} V_i$ s.t.

$p|_{V_i} : V_i \to U$ is a homeo.



Rmk: If $U \subseteq X$ is evenly covered, and $V \subseteq U$, then V is also evenly covered:

$$p^{-1}(V) = p^{-1}(V) \cap p^{-1}(U) = \coprod_{i \in I} \left(p^{-1}(V) \cap V_i\right)$$

$p : V_i \to U$ is a homeo $\Rightarrow$ $p : V_i \cap p^{-1}(V) \to V$ is also a homeo.

Def: $p : \tilde{X} \to X$ is a <u>covering map</u> if any $x \in X$ has an evenly covered nbh.

$p^{-1}(x) \subseteq \tilde{X}$ is the <u>fiber of x</u>. $\# p^{-1}(x)$ is the <u>degree of p at x</u>.

Def: $f : Y \to Z$ is a <u>local homeo</u> if $\forall\, y \in Y$ $\exists$ $U \ni y$ open, $V \ni f(y)$ open s.t. $f : U \to V$ is a homeo.

Lemma (lecture 18): X is connected, $p: \tilde{X} \to X$ cont.

(1) If p is a covering map, then it is a local homeo and the degree of p is constant.

(2) If p is a local homeo, and $p^{-1}(x)$ is finite $\forall x$, then p is a covering map.

First exercises

9.3: $p: \mathbb{C}^* \to \mathbb{C}^*$; $z \mapsto z^n$. This is a covering map of degree n.

Pf: $\forall z \in \mathbb{C}^*$ $\# p^{-1}(z) = \# \{ \zeta : \underbrace{\zeta^n - z}_{\hookrightarrow \text{ exactly } n \text{ distinct roots } (z \neq 0)} = 0 \} = n.$

By lemma (2), left to show: p is a local homeo:

$p: \mathbb{C}^* \to \mathbb{C} : z \mapsto z^n$ is holomorphic, $p'(z) = n z^{n-1} \neq 0$ $\forall z \in \mathbb{C}^*$.

Inverse function thm, p has a local holomorphic inverse $\Rightarrow$ p is a local homeo $\blacksquare$

9.4: $p: \tilde{X} \to X$ cont.

(i) p is a covering map $\Longleftrightarrow$ $\exists$ an open cover of X by ev. cov. sets.

(ii) p is a covering map $\Longleftrightarrow$ $\exists$ a basis of X by ev. cov. sets.

(ii) p is a covering map $\iff \exists$ a basis of X by ev. cov. sets.

(iii) Covering maps are open.

(i) By definition ...

(ii) "$\Longleftarrow$" A basis is an open cover.

"$\Longrightarrow$" $\mathcal{B} := \{$ evenly covered open sets $\subseteq X\}$. is a basis

$$\bigcup_{B \in \mathcal{B}} B = X \quad \text{by (i).}$$

$B_1, B_2 \in \mathcal{B}$, $\quad B_1 \cap B_2$ is open and evenly covered $(\subseteq B_1)$

$$\Longrightarrow \in \mathcal{B}.$$

(iii) Let $\mathcal{B}$ be as above.

$$\begin{array}{ccc} \tilde{X} & \xrightarrow{p} & X \\ \cup\!| & & \cup\!| \\ \tilde{U} & \longrightarrow & p(\tilde{U}) \end{array}$$

$$p(\tilde{U}) = p(\tilde{U}) \cap \Big(\bigcup_{B \in \mathcal{B}} B\Big) =$$

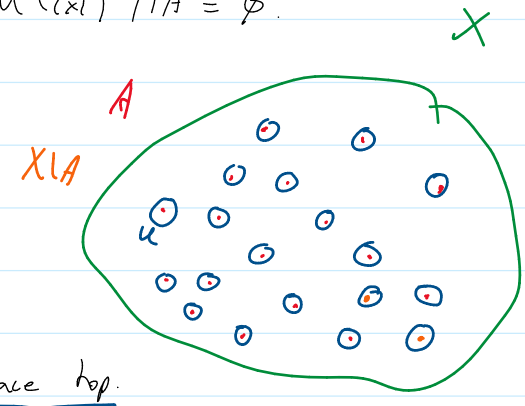
$$= \bigcup_{B \in \mathcal{B}} \underbrace{p(\tilde{U}) \cap B}_{\hookrightarrow \text{ to show: this is open}}$$

$$p^{-1}(B) = \coprod V_i \qquad p(\tilde{U}) \cap B = p(\tilde{U} \cap \coprod V_i) = \bigcup p(\tilde{U} \cap V_i)$$

But $\tilde{U} \cap V_i \subseteq V_i$ is open, $p|_{V_i}$ is a homeo $\Rightarrow p(\tilde{U} \cap V_i)$ is open. $\blacksquare$

Discrete subsets, ex. 9.7

Def: X any top. space, $A \subseteq X$ is a <u>discrete subset</u> if $\forall\ x \in X\ \exists$ a nbh $U \ni x$ s.t. $(U \setminus \{x\}) \cap A = \emptyset$.



In other words:

(a) $\forall\ x \in A\ \exists\ U \ni x : U \cap A = \{x\}$

(b) $\forall\ x \notin A\ \exists\ U \ni x : U \cap A = \emptyset$.

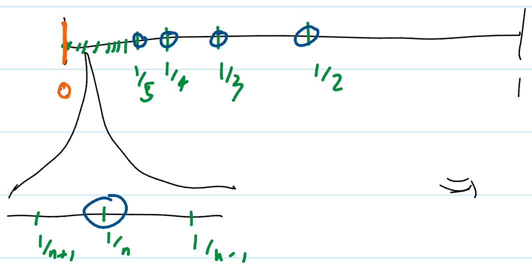
(a) $\iff$ <u>A is a discrete topological space w/ subspace top.</u>

L (a) $\iff$ $\{x\}$ is open in $(A, \mathcal{T}_A)$ $\forall\ x \in A$ $\iff$ $(A, \mathcal{T}_A)$ is discrete.

(b) $\iff$ <u>A is closed in X.</u>

L (b) $\iff$ $\forall\ x \notin A\ \exists\ U \ni x : x \in U \subseteq X \setminus A$ $\iff$ $X \setminus A$ is open.

EX:



$A = \{ \frac{1}{n} : n \geq 1 \}$.

$\Rightarrow$ A satisfies (a) ✓, <u>but</u>

A does <u>not</u> satisfy (b)!

COR: A discrete subset of a compact space X is finite.

Pf. $A \subseteq X$ discrete. A is closed (b), so $(A, \mathcal{T}_A)$ is compact.

Also $(A, \mathcal{T}_A)$ is discrete (a). So A is finite.

Ex 9.7: $p : \tilde{X} \to X$ covering map, X Hausdorff. Then $\forall\ x \in X$,

$p^{-1}(x) \subseteq \tilde{X}$ is a discrete subset. In particular, $\tilde{X}$ compact $\Rightarrow p^{-1}(x)$ finite.

Ex 9.7: $p: \tilde{X} \to X$ covering map, X Hausdorff, then $\forall\, x \in X$, $p^{-1}(x) \subseteq \tilde{X}$ is a discrete subset. In particular, $\tilde{X}$ compact $\Rightarrow p^{-1}(x)$ finite.

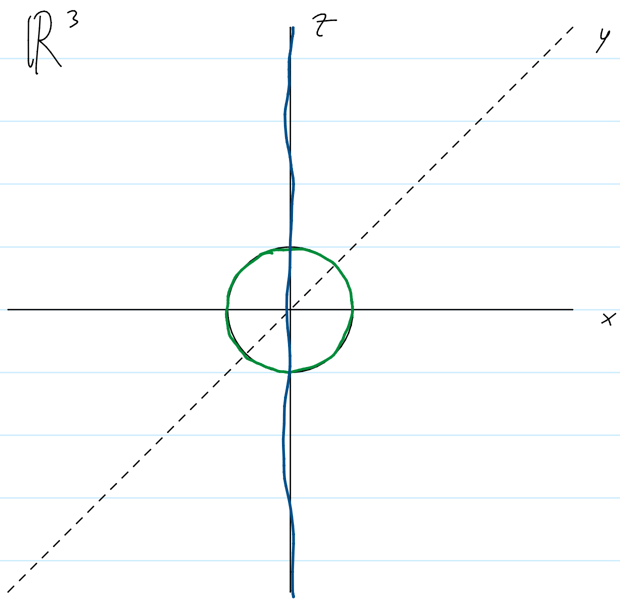
Pf: (a): $y \in p^{-1}(x)$. Want: $V \ni y$ s.t. $V \cap p^{-1}(x) = \{y\}$.

Let $U \ni x$ open, evenly covered. $p^{-1}(U) = \coprod_{i \in I} V_i$. $p|_{V_i}: V_i \to U$ is a homeo $\Rightarrow \exists!\, i_0 \in I$ s.t. $y \in V_{i_0}$ open.

$V_{i_0} \cap p^{-1}(x) = \{y\}$ b/c $p: V_{i_0} \to U$ is injective.

(b): X is Hausdorff $\Rightarrow \{x\}$ is closed $\overset{p \text{ cont.}}{\Rightarrow} p^{-1}(x)$ is closed.

Challenge problem



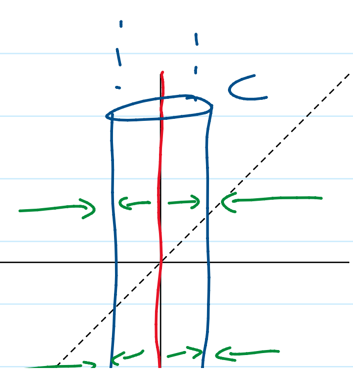
$\gamma = \{x^2 + y^2 = 1,\ z = 0\}$

$\ell = \{x = y = 0\}$

Q: Show that $\boxed{\mathbb{R}^3 \setminus \gamma \not\cong \mathbb{R}^3 \setminus (\gamma \cup \ell)}$

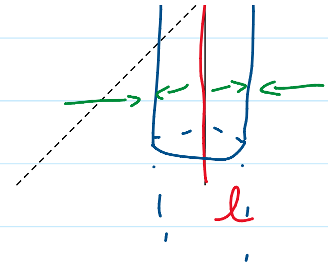
Idea: show that π_1's are different.

① $\mathbb{R}^3 \setminus \ell$



- Retract everything to an ∞ cylinder.

$\mathbb{R}^3 \setminus \ell = \{(\theta, r, z) : r > 0\}$



$\mathbb{R}^3 \setminus \ell = \{ \underbrace{(\theta, r, z)}_{\text{cylindrical coordinates}} : r > 0 \}$

The retraction is: $(\theta, r, z) \longmapsto (\theta, 1, z)$

Def. $A \subseteq X$ is a deformation retract of X if $\exists\ H : X \times I \to X$

$H(-, 0) = id_X, \quad H(X, 1) \subseteq A, \quad H(a, t) = a \quad \forall\ a \in A.$

Note: A and X are homotopic.

$A \xhookrightarrow{incl.} X$

$A \xleftarrow{H(-,1)} X$

$H : \mathbb{R}^3 \setminus \ell \times I \longrightarrow \mathbb{R}^3 \setminus \ell$

is a def. retr. onto C

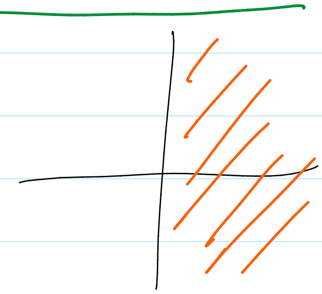
$$\boxed{H((\theta, r, z), t) = (\theta, r(1-t) + t, z).}$$

homotopic

$\Rightarrow \quad \mathbb{R}^3 \setminus \ell \simeq C \cong S^1 \times \mathbb{R}$

$\pi_1(\mathbb{R}^3 \setminus \ell) \cong \pi_1(S^1 \times \mathbb{R}) \cong \pi_1(S^1) \times \underbrace{\cancel{\pi_1(\mathbb{R})}}_{=\{1\}} \cong \mathbb{Z}.$

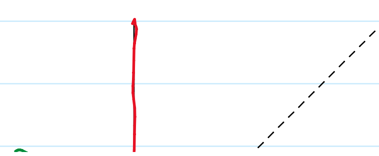
- Let H be an open half-plane $\subseteq \mathbb{R}^2$



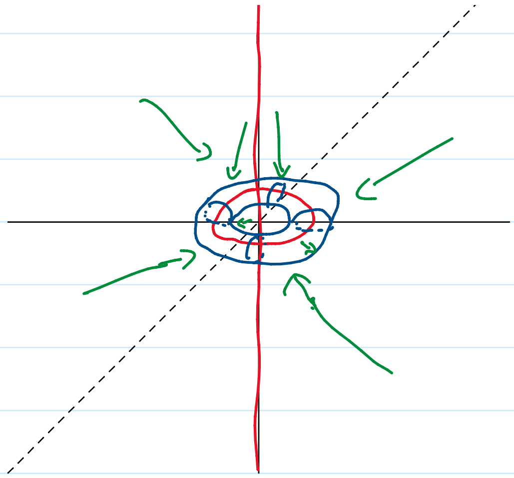
$\Rightarrow \quad \mathbb{R}^3 \setminus \ell \cong H \times S^1$

$\pi_1(\mathbb{R}^3 \setminus \ell) \cong \cancel{\pi_1(H)} \times \pi_1(S^1) \cong \mathbb{Z}.$

② $\mathbb{R}^3 \setminus (\ell \cup \gamma)$



② $\mathbb{R}^3 \setminus (\ell \cup \gamma)$



- Retract on a torus.

$\Rightarrow$ $\mathbb{R}^3 \setminus (\ell \cup \gamma) \simeq \mathbb{T}^2 \cong S^1 \times S^1$

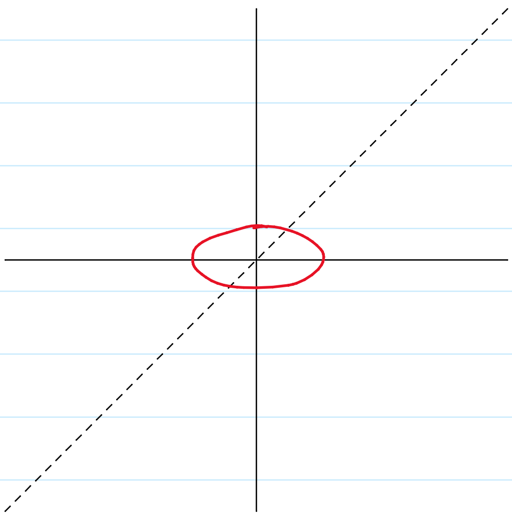
$\Rightarrow$ $\pi_1(\mathbb{R}^3 \setminus (\ell \cup \gamma)) \cong \mathbb{Z} \times \mathbb{Z}$.

- $\mathbb{R}^3 \setminus (\ell \cup \gamma) \cong \underbrace{H \setminus \{pt\}}_{\simeq S^1} \times S^1$

$\Rightarrow$ $\pi_1(\mathbb{R}^3 \setminus (\ell \cup \gamma)) \cong \pi_1(H \setminus \{pt\}) \times \pi_1(S^1) \cong \mathbb{Z} \times \mathbb{Z}$.

Therefore: $\boxed{\mathbb{R}^3 \setminus \ell \cup \gamma \not\cong \mathbb{R}^3 \setminus \ell.}$

③ $\mathbb{R}^3 \setminus \gamma$:



$i: \mathbb{R}^3 \setminus \{0\} \longrightarrow \mathbb{R}^3 \setminus \{0\}$

$x = \underset{\mathbb{R}_{>0}}{\underset{\Uparrow}{r}} \cdot \underset{S^2}{\underset{\Uparrow}{v}} \longmapsto r^{-1} v$

i(circle touching 0) = (infinite line not touching 0)

touch 0

$\Rightarrow$ $\boxed{i: \mathbb{R}^3 \setminus \gamma \xrightarrow{\cong} \mathbb{R}^3 \setminus (\ell \cup \{0\})}$