Plan:
- Common mistakes / confusions on §58
- Reminders on covering maps
- §58, 9.3, 9.4
- 9.7 + discrete subsets
- Challenge problem

Common mistakes / confusions on §58

1. Difference between free homotopy and based homotopy.

2. $f, g : X \rightarrow Y$ are homotopic if $\exists H : X \times I \rightarrow Y$ s.t.

   $H(-, 0) = f$, \hspace{1cm} $H(-, 1) = g$.

3. Two paths $y, f : I \rightarrow X$ are (freely) homotopic if

   $\exists H : I \times I \rightarrow X$ s.t. $H(-, 0) = y$, \hspace{0.5cm} $H(-, 1) = f$.

   and $y, f : I \rightarrow X$ s.t. $y(0) = f(0)$, \hspace{0.5cm} $y(1) = f(1)$.

   They are homotopic as paths / homotopic relative to their endpoints.

   if $\exists H : I \times I \rightarrow X$ s.t. 1) $H(-, 0) = y$, \hspace{0.5cm} $H(-, 1) = f$

   2) $H(0, -) = y$, \hspace{0.5cm} $H(1, -) = f$.

   $H(t, 0) = y(t)$, \hspace{0.5cm} $H(t, 1) = f(t)$.
EX:

Then $\gamma$ is freely homotopic to $\delta$
(both homotopic to $c : I \to S^1$)

but not homotopic rel. to their endpoints.

Indeed $\gamma \simeq \delta \Rightarrow \left[ \gamma \ast \delta^{-1} \right] = \text{id} \in \pi_1(S^1)$

However $\gamma \ast \delta^{-1}$ is a generator of $\pi_1(S^1) \cong \mathbb{Z}$.

(2) Homotopy of constant maps

Suggestion: When talking about a constant map $X \to Y: x \mapsto y_0$,

Write $C_{y_0}$, not $y_0$.

I have seen the following mistakes: $f : X \to Y$

$$f(x) = g(x) \quad \forall x \quad \Rightarrow \quad f = g$$

In a path-connected space $Y$, all constant maps $X \to Y$ are homotopic.

$\gamma \neq \gamma_0$ but $\delta'$ is path-connected, so $\gamma \simeq \delta$. 
Reminder on covering maps

Def. $U \subseteq X$ is evenly covered if $p^{-1}(U) = \bigcup_{i \in I} V_i$ s.t.

$p|_{V_i} : V_i \rightarrow U$ is a homeo.

Remark: If $U \subseteq X$ is evenly covered, and $V \subseteq U$, then $V$ is also evenly covered.

$p^{-1}(V) = p^{-1}(U) \cap p^{-1}(V) = \bigcup_{i \in I} (p^{-1}(V) \cap V_i)$

$p : V_i \rightarrow U$ is a homeo $\Rightarrow$ $p : V_i \cap p^{-1}(V) \rightarrow V$ is also a homeo.

Def: $p : \tilde{X} \rightarrow X$ is a covering map if any $x \in X$ has an evenly covered nbh.

$p^{-1}(x) \subseteq \tilde{X}$ is the fiber of $x$. $\# p^{-1}(x)$ is the degree of $p$ at $x$.

Def: $f : Y \rightarrow Z$ is a local homeo if $\forall y \in Y \exists U \ni y$ open s.t. $f : U \rightarrow V$ is a homeo.
Lemma (lecture 18): \( X \) is connected, \( p: X \rightarrow X \) cont.

(i) If \( p \) is a covering map, then it is a local homeo and the degree of \( p \) is constant.

(ii) If \( p \) is a local homeo and \( p^{-1}(x) \) is finite \( \forall x \), then \( p \) is a covering map.

First exercises

9.3: \( p: \mathbb{C}^* \rightarrow \mathbb{C}^*: z \mapsto z^n \). This is a covering map of degree \( n \).

Proof: \( \forall z \in \mathbb{C}^* \), \( \# p^{-1}(z) = n \). \( \sum_{k=0}^{n-1} z^k = 0 \iff n \neq 0 \) \iff \( z \neq 0 \), exactly \( n \) distinct roots \( z \neq 0 \).

By lemma (2), left to show: \( p \) is a local homeo.

\( p: \mathbb{C}^* \rightarrow \mathbb{C}^*: z \mapsto z^n \) is holomorphic, \( p^{-1}(z) = n z^{-n-1} \neq 0 \forall z \neq 0 \).

Inverse function theorem: \( p \) has a local holomorphic inverse \( \iff p \) is a local homeo.

9.4: \( p: X \rightarrow X \) cont.

(i) \( p \) is a covering map \( \iff \) \( \exists \) an open cover of \( X \) by ev. cov. sets.

(ii) \( p \) is a covering map \( \iff \exists \) a basis of \( X \) by ev. cov. sets.
(iii) \(p\) is a covering map \(\iff\) \( \exists \) a basis of \(X\) by ev. cov. sets.

By definition, 

(iii) \(\Rightarrow\) A basis is an open cover.

\(\Rightarrow\) \(\exists B = \{\text{evenly covered open sets} \subseteq X \mid \text{is a basis}\}

\(\bigcup_{b \in B} b = X \) by (ii).

\(b_i, b_j \in B, b_i \cap b_j \text{ is open and evenly covered} (\subseteq B_i)\)

\(\Rightarrow c \in B.\)

Let \(\tilde{B}\) be as above.

\(\tilde{X} \xrightarrow{p} X\)

\(p(\tilde{u}) = p(\tilde{u}) \cap \left( \bigcup_{b \in B} b \right) = \tilde{u} \xrightarrow{\text{to show: this is open}} p(\tilde{u})\)

\(\bigcup_{b \in B} p(\tilde{u}) \cap b\)

But \(\tilde{u} \cap V_i \subseteq V_i \text{ is open, } p(V_i \text{ is a homeo} \Rightarrow p(\tilde{u} \cap V_i) \text{ is open.}\)

\text{Discrete subsets, ex. 9.7}
**Def:** \( X \) any top. space, \( A \subseteq X \) is a **discrete subset** if
\[ \forall x \in X \exists U \ni x \text{ s.t. } U \cap A = \emptyset \]

In other words:
\[
(a) \forall x \in A \exists U_x : U \cap A = \emptyset \\
(b) \forall x \notin A \exists U_x : U \cap A = \emptyset
\]

(a) \( \Rightarrow \) \( A \) is a discrete topological space w/ subspace top.

\[
(\text{a}) \Rightarrow \text{\( \emptyset \times I \) is open in \((A, J_A)\) \( \forall x \in A \Rightarrow (A, J_A) \) is discrete.}
\]

(b) \( \Leftarrow \) \( A \) is closed in \( X \).

\[
(\text{b}) \Leftarrow \text{\( \forall x \notin A \exists U_x : x \in U \subseteq X \setminus A \Rightarrow X \setminus A \) is open.}
\]

**EX:** \( \triangle \)

\( A = \{ \frac{1}{n} : n \geq 1 \} \)

\( \Rightarrow \) \( A \) satisfies (\( a \)) \text{ but}

\( A \) does not satisfy (\( b \))!

**COR:** A discrete subset of a compact space \( X \) is finite.

**Pt.** \( A \subseteq X \) discrete, \( A \) is closed (\( b \)), so \((A, J_A)\) is compact.

Also \((A, J_A)\) is discrete (\( a \)), so \( A \) is finite.

**Ex 9.7:** \( p : \tilde{X} \to X \) covering map, \( X \) Hausdorff. Then \( \forall x \in X \)

\[ p^{-1}(x) \subseteq \tilde{X} \text{ is a discrete subset. In particular, } \tilde{X} \text{ compact } \Rightarrow p^{-1}(x) \text{ finite.} \]
Example: \( p \colon X \to \bar{X} \) covering map, \( X \) Hausdorff, then \( \forall x \in X \), \( p^{-1}(x) \subseteq \bar{X} \) is a discrete subset. In particular, \( \bar{X} \) compact \( \Rightarrow p^{-1}(x) \) finite.

**Proof:** (a): \( y \in p^{-1}(x) \). Want: \( \forall y \) s.t. \( \forall \cap p^{-1}(x) = \{ y \} \).

Let \( U \ni x \) open, evenly covered. \( p^{-1}(U) = \bigsqcup_{i=1}^{n} V_i \). \( p \colon V_i \to \bar{U} \) is a homeo \( \Rightarrow \exists ! i_0 \in I \) s.t. \( y \in V_{i_0} \) open.

\( V_{i_0} \cap p^{-1}(x) = \{ y \} \) \( \forall \) \( \in \) \( p \) cont.

(b): \( X \) is Hausdorff \( \Rightarrow \exists ! x \) is closed \( \Rightarrow \forall \cap p^{-1}(x) \) is closed.

**Challenge problem**

\( Y = \{ x^2 + y^2 = 1 \} \)

\( L = \{ x = y = 0 \} \)

Q: Show that \( \mathbb{R}^3 \setminus Y \neq \mathbb{R}^3 \setminus (Y \cup L) \)

Idea: show that \( \sigma_1, \sigma_2 \) are different

1. \( \mathbb{R}^3 \setminus L \)

   - Contract everything to an \( \infty \) cylinder.

   \( \mathbb{R}^3 \setminus L = \{ (\theta, r, z) : r > 0 \} \)
\[ \mathbb{R}^3 \setminus \mathcal{L} = \{ (0, r, z) : r > 0 \} \]

under cylindrical coordinates

The retraction is \( (0, r, z) \to (0, 1, z) \)

**Def:** \( A \subset X \) is a deformation retract of \( X \) if \( \exists H : X \times I \to X \)

\[ H(-, 0) = \text{id}_X, \quad H(x, 1) \in A, \quad H(x, t) = a \quad \forall x \in A. \]

**Note:** \( A \) and \( X \) are homotopic.

\[ A \subset \xrightarrow{\text{homotopy}} X \]

\[ H(\cdot, 1) \]

\[ H: \mathbb{R}^3 \setminus \mathcal{L} \times I \to \mathbb{R}^3 \setminus \mathcal{L} \]

is a def. rem. onto \( \mathcal{C} \)

\[ H((0, r, z), t) = (0, r(t-t) + t, z). \]

\[ \mathbb{R}^3 \setminus \mathcal{L} \cong C \cong S^1 \times \mathbb{R} \]

\[ \pi_1(\mathbb{R}^3 \setminus \mathcal{L}) \cong \pi_1(S^1 \times \mathbb{R}) \cong \pi_1(S^1) \cong \mathbb{Z}. \]

Let \( H \) be an open half-plane \( \in \mathbb{R}^2 \)

\[ \mathbb{R}^3 \setminus \mathcal{L} \cong H \times S^1 \]

\[ \pi_1(\mathbb{R}^3 \setminus \mathcal{L}) \cong \pi_1(H \times S^1) \cong \mathbb{Z}. \]
① $\mathbb{R}^3 \setminus (L \cup Y)$

- Retract on a torus.

\[ \mathbb{R}^3 \setminus (L \cup Y) \cong \mathbb{R}^2 \cong S^1 \times S^1 \]

\[ \therefore \quad \pi_1(\mathbb{R}^3 \setminus (L \cup Y)) \cong \mathbb{Z} \times \mathbb{Z} \]

\[ \text{Therefore:} \quad \mathbb{R}^3 \setminus (L \cup Y) \neq \mathbb{R}^3 \setminus L. \]

② $\mathbb{R}^3 \setminus X$

$\varphi : \mathbb{R}^3 \setminus \{0\} \to \mathbb{R}^3 \setminus \{0\}$

\[ x = r \cdot \nu \mapsto r^{-\nu} \quad \text{where} \quad (r, \nu) \in \mathbb{R}^2 \times S^1 \]

$\varphi$ (circle touching 0) = (infinite line not touching 0)

\[ \therefore \quad \varphi : \mathbb{R}^3 \setminus Y \cong \mathbb{R}^3 \setminus (L \cup \{0\}) \]