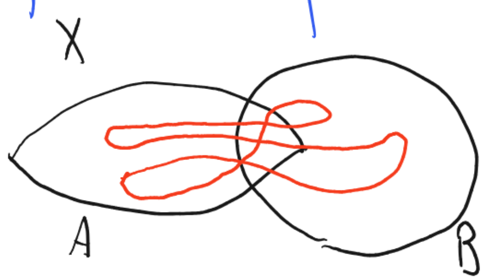
Topology exercise class 11/05

Today:

- Recall Seifert - van Kampen
- Applications of SvK (ex 10.1 and 10.7)
- Representing surfaces with polygons (ex 10.5)
- Exercises 10. 8 and 10.9
- (if time allows) fundamental group of graphs

Seifert - van Kampen



Theorem: let X be path-connected space. $A, B \subseteq X$ opens sets s.t. $X = A \cup B$, and $A, B, A \cap B$ are all path-connected.

Then there is surjective homomorphism

$\Phi : \pi_1(A) * \pi_1(B) \longrightarrow \pi_1(X)$. Moreover its Ker is generated by $\{ \varphi(w) \psi(w)^{-1} : w \in \pi_1(A \cap B) \}$

$\varphi : \pi_1(A \cap B) \to \pi_1(A)$, $\psi : \pi_1(A \cap B) \to \pi_1(B)$ induced by inclusions.

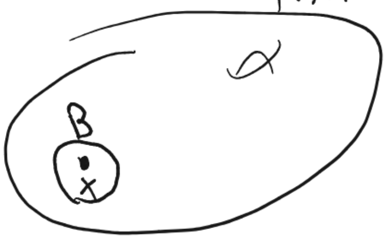
Remark: SvK is the reason why π_1 is easy.

Exercise 10.1

X be top. manifold of $\dim \geq 3$. $x \in X$, then
$\pi_1(X) \cong \pi_1(X \setminus \{x\})$.

Rmk: If $X = \mathbb{R}^2$, $x = (0,0)$ this no true!

Idea: Use SvK to relate $\pi_1(X)$ with $\pi_1(A)$ where
$A = X \setminus \{x\}$



Take B open nbh of x
homeomorphic to a ball (of dim $n = \dim X$).

$A \cup B = X$

$A \cap B = B \setminus \{x\} \cong$ Ball $\setminus$ point

$\simeq S^{n-1}$

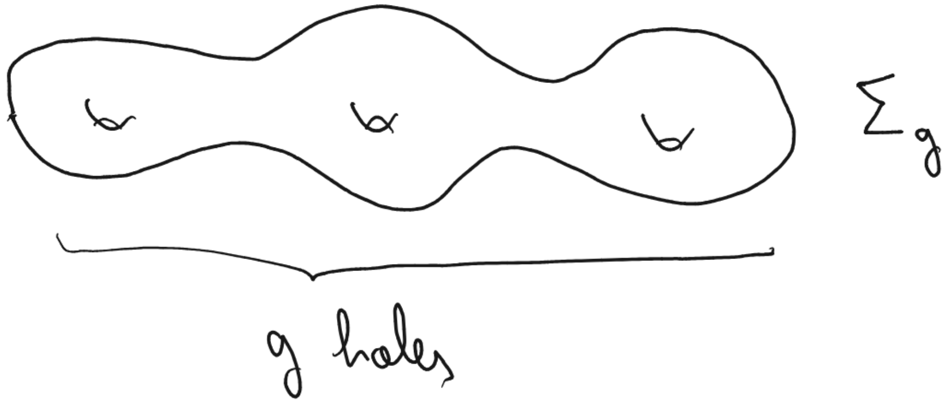
S^{n-1}

SvK gives homomorphism:

$\Phi : \pi_1(X \setminus \{x\}) * \underset{\{1\}}{\underset{\shortparallel}{\pi_1(B)}} \to \pi_1(X)$ surjective.

$\ker \Phi = \{1\}$ generated by $\{\varphi(w) : w \in \pi_1(A \cap B)\}$ $\cong \pi_1(S^{n-1}) \cong \{1\}$ $n \geq 3$. $\square$

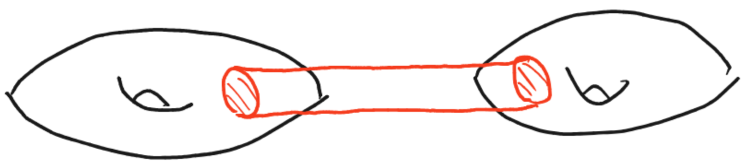
Genus g surface (ex 10.5 and 10.7)



In particular $\Sigma_1 = T$

How to formally define Σ_g?

① Connected sum



$T \# T \cong \Sigma_2$

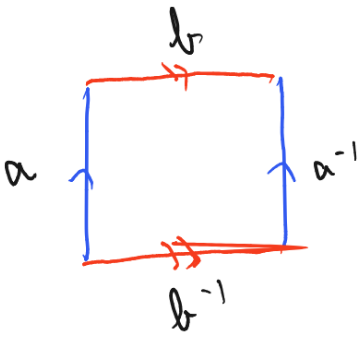
Def: X, Y be top manifolds of same dim.
Construct $X \# Y$ by removing open balls from each and gluing along the boundary.

Fact (hard!): $X \# Y$ doesn't depend on the balls we chose neither on the glue.

② Using polytopes to describe surfaces.

Torus: $T \cong [0,1]\times[0,1]/\sim$

$(x,0)\sim(x,1)$ and $(0,y)\sim(1,y)$



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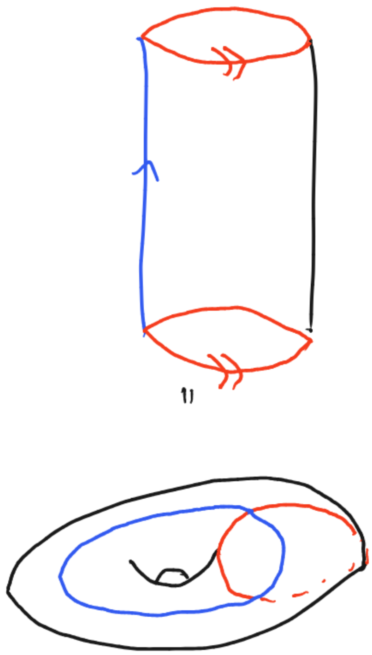
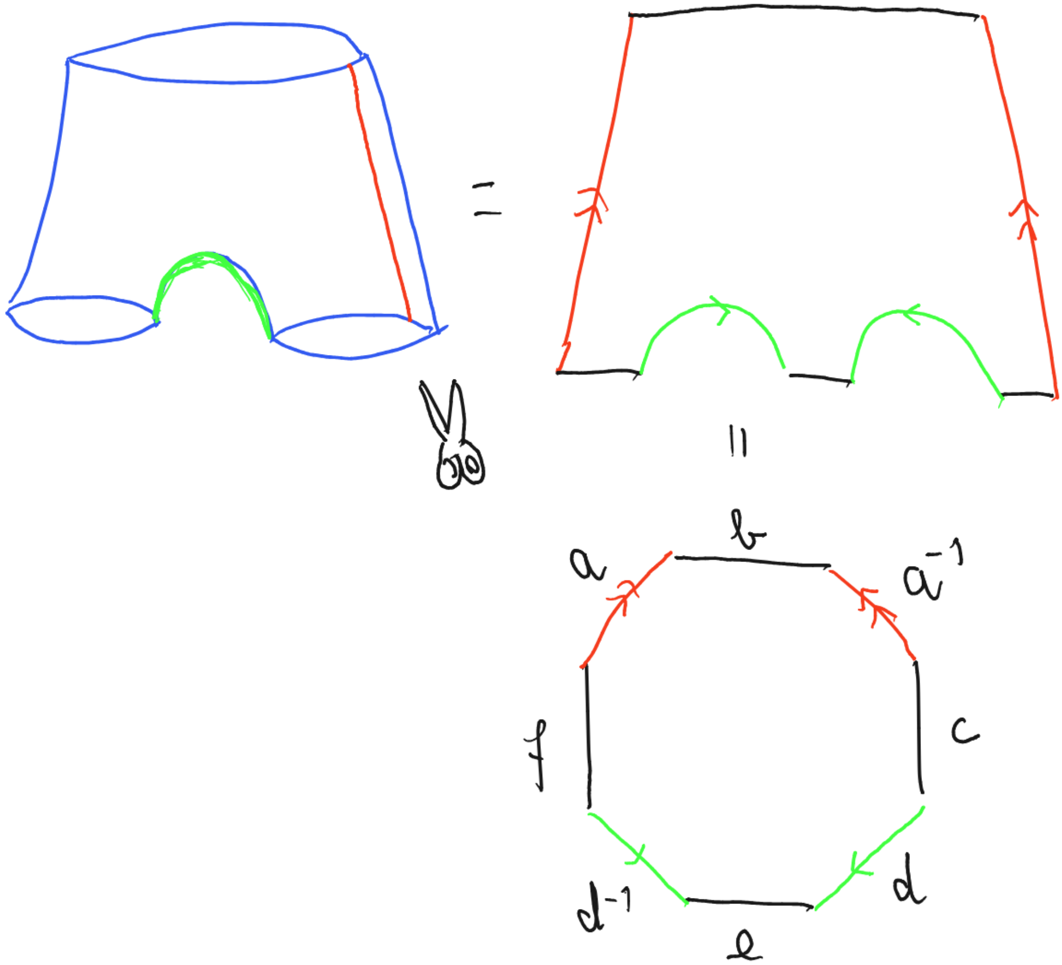


$\downarrow$

$a\,b\,a^{-1}b^{-1}$

Pair of Pants

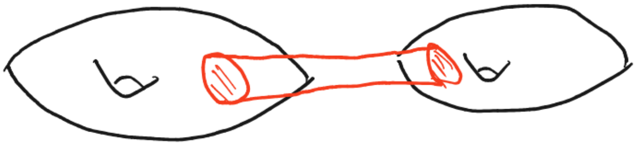


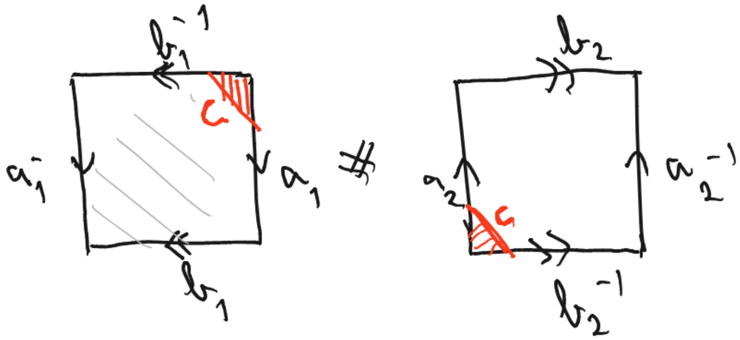
$\downarrow$

$a\, b\, a^{-1}\, c\, d\, e\, d^{-1}\, f$

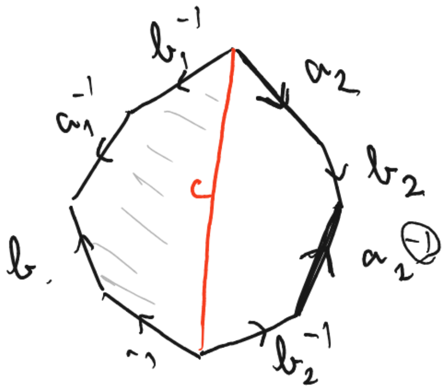
Genus g surface

$g = 2$





$\cong$

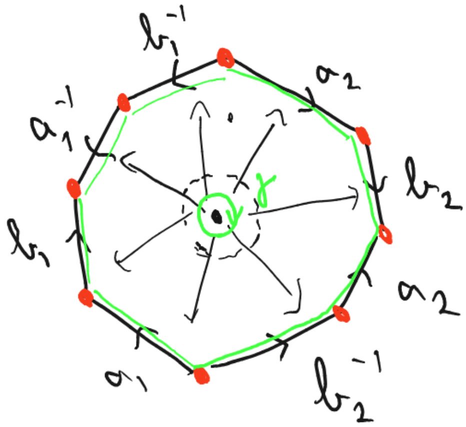


$\rightsquigarrow a_1 b_1 a_1^{-1} b_1^{-1} a_2 b_2 a_2^{-1} b_2^{-1}$

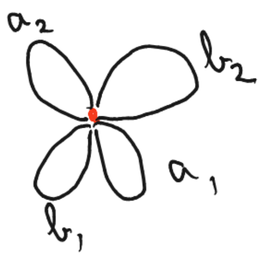
Genus g surface $\rightsquigarrow a_1 b_1 a_1^{-1} b_1^{-1} \dots a_g b_g a_g^{-1} b_g^{-1}$

Compute $\pi_1(\Sigma_g)$, $\pi_1(\Sigma_g \setminus \{pt\})$

10.7   10.5.



Can deformation retract $\Sigma_g \setminus \{pt\}$ to the "boundary".



$$\Sigma_g \setminus \{pt\} \simeq \underbrace{S^1 \vee \dots \vee S^1}_{2g}$$

$$\Downarrow$$

$$\pi_1(\Sigma_g \setminus \{pt\}) \cong \underbrace{\mathbb{Z} * \dots * \mathbb{Z}}_{2g}$$

$$= \langle a_1, \dots, a_g, b_1, \dots, b_g \rangle$$

$\pi_1(\Sigma_g)$

SvK gives surjective map

$$\Phi : \pi_1(\Sigma_g \setminus \{pt\}) \to \pi_1(\Sigma_g)$$

$$\ker \Phi \text{ is generated by } \{\varphi(w) : w \in \pi_1(\mathrm{Ball} \setminus \{pt\})\}$$

$$[\gamma] \in \pi_1(\mathrm{Ball} \setminus \{pt\}) \cong \pi_1(S^1) \cong \mathbb{Z}$$

$\cancel{\Phi : \pi_1(\Sigma_g}$

$$\begin{aligned}
\varphi : \pi_1(S^1) &\longrightarrow \pi_1(\Sigma_g \setminus \{pt\}) \\
\mathbb{Z} &\longrightarrow \langle a_1, \dots, a_g, b_1, \dots, b_g \rangle \\
[\gamma] = 1 &\longmapsto \underbrace{a_1 b_1 a_1^{-1} b_1^{-1} \dots a_g b_g a_g^{-1} b_g^{-1}}
\end{aligned}$$

SvK: $\pi_1(\Sigma_g) \cong \pi_1(\Sigma_g \setminus \{pt\}) / \ker \Phi$

$$\cong \langle a_1, \dots, a_g, b_1, \dots, b_g \mid \qquad = 1 \rangle$$

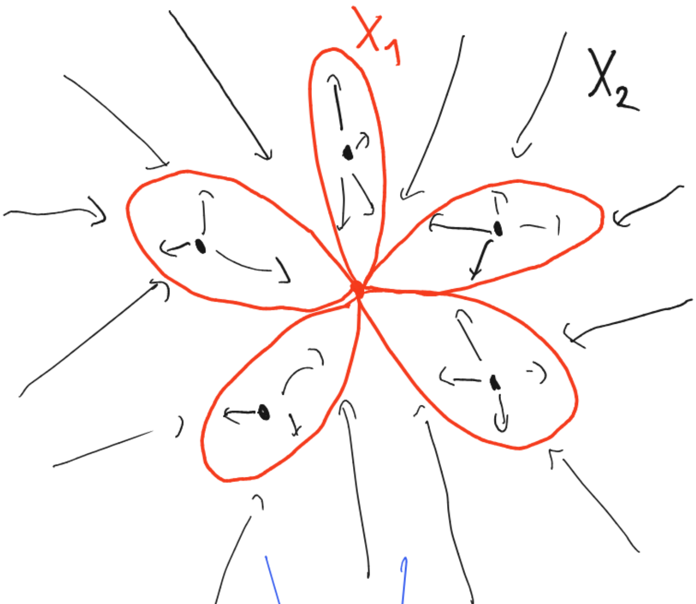
In particular $\pi_1(T) \cong \langle a, b \mid a b a^{-1} b^{-1} = 1 \rangle$

$$\cong \mathbb{Z} \oplus \mathbb{Z}.$$

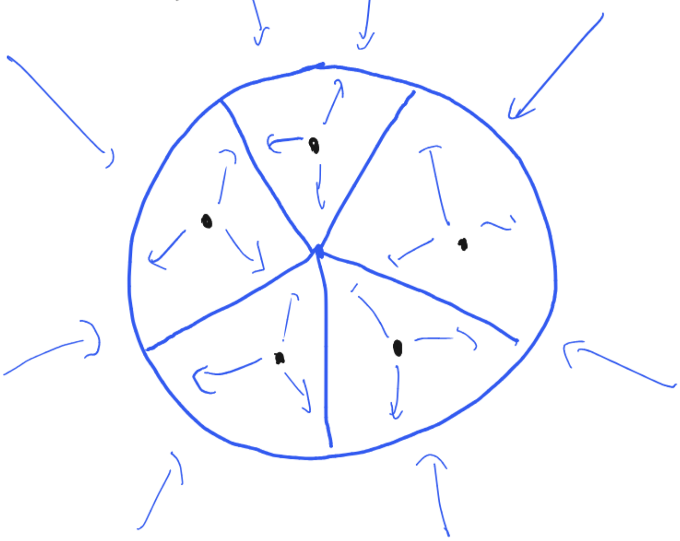
Exercise 10.8 and 10.9

$k \geq 1$

- $X_1 = \underbrace{S^1 \vee \dots \vee S^1}_{k}$
- $X_2 = \mathbb{R}^2 \setminus \{k \text{ points}\}$
- $X_3 = S^1 \cup \{k \text{ radii}\}$



X_2 deformation retracts to X_1



Ex 10.9

$\pi_1(\mathbb{R}^n \setminus \{k \text{ points}\}) \quad n \geq 2.$

- $n = 2$

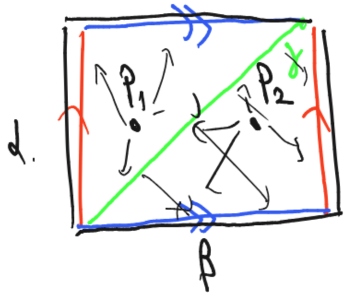
$$\pi_1(\mathbb{R}^n \setminus \{k \text{ points}\}) \simeq \pi_1(\underbrace{S^1 \vee \dots \vee S^1}_{k}) \cong \underbrace{\mathbb{Z} * \dots * \mathbb{Z}}_{k}.$$

- $n \geq 3$

$$\pi_1(\mathbb{R}^n \setminus \{k \text{ pts}\}) \overset{\text{inductively w/ 10.1}}{\cong} \pi_1(\mathbb{R}^n) \cong \{1\}.$$

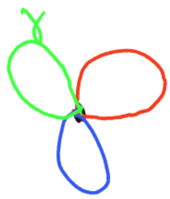
Question in the forum

Computing $\pi_1(T)$ with SvK with $A = T\setminus\{p_1\}$, $B = T\setminus\{p_2\}$



$$\pi_1(A) = \langle \alpha_1, \beta_1 \rangle$$
$$\pi_1(B) = \langle \alpha_2, \beta_2 \rangle$$

$$\pi_1(A\cap B) = \langle \alpha, \beta, \gamma \rangle$$

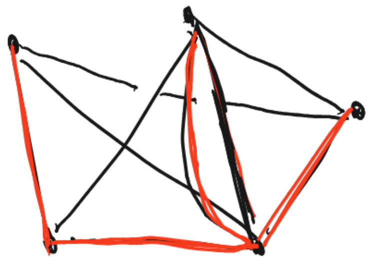


$$\begin{array}{lcl} \pi_1(A\cap B) \to \pi_1(A) & & \pi_1(A\cap B) \to \pi_1(B) \\ \alpha \longmapsto \alpha_1 & \leftarrow\text{-}\text{-}\to & \alpha \longmapsto \alpha_2 \\ \beta \longmapsto \beta_1 & \leftarrow\text{-}\text{-}\to & \beta \longmapsto \beta_2 \\ \gamma \longmapsto \beta_1\alpha_1 & \leftarrow\text{-}\text{-}\to & \gamma \longmapsto \alpha_2\beta_2 \end{array}$$

$$\pi_1(T) = \langle \alpha_1, \beta_1, \alpha_2, \beta_2 \ \cancel{\gamma} \mid \alpha_1 = \cancel{\beta_1}\alpha_2,\ \beta_1 = \beta_2,\ \beta_1\alpha_1 = \alpha_2\beta_2 \rangle$$
$$\| $$
$$\langle \alpha_1, \cancel{\ } \beta_1 \mid \beta_1\alpha_1 = \alpha_1\beta_1 \rangle \cong \mathbb{Z} \oplus \mathbb{Z}$$

π_1 (graphs) : ex 10.10

π_1 ()

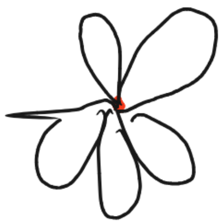


G = graph

T = spanning tree

$e(T) = V - 1$

$$\begin{array}{l} \pi_1(G/T) = \pi_1(G) \\ \quad\;\; \| \\ \underbrace{\mathbb{Z} * \cdots * \mathbb{Z}}_{E - V + 1} \end{array}$$



$$\underbrace{S^1_1 \vee S^1 \vee \cdots \vee S^1}_{E - V + 1}$$