Topology exercise class 18/05

Today: • Small remark on SvK
• Discuss problems 11.2, 11.3, 11.6, 11.7, 11.8

Remark about SvK

$$X = A \cup B \rightsquigarrow \Phi : \pi_1(A) * \pi_1(B) \to \pi_1(X) \quad \text{surjective}$$

Ker Φ → always normal!

Ker Φ is generated as normal subgroup by

$$\{\varphi(w)\psi(w)^{-1} : w \in \pi_1(A \cap B)\}$$

$$\varphi : \pi_1(A \cap B) \to \pi_1(A) \quad , \quad \psi : \pi_1(A \cap B) \to \pi_1(B)$$

i.e. ker Φ is the smallest normal subg. containing

Ex: $\Phi : \pi_1(S^1 \vee S^1) \to \pi_1(T)$

$$\begin{array}{ccc} \varphi : \pi_1(A \cap B) & \longrightarrow & \pi_1(A) \\ \wr\Vert & & \wr\Vert \\ \mathbb{Z} & & \mathbb{Z} * \mathbb{Z} \\ 1 & \longmapsto & aba^{-1}b^{-1} \end{array}$$

$$\overset{\cong \mathbb{Z}}{\ker \Phi} \neq \{(aba^{-1}b^{-1})^k : k \in \mathbb{Z}\}$$

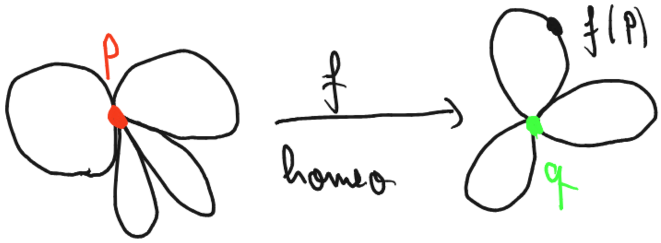
$$\underbrace{a(\overset{=1}{aba^{-1}b^{-1}})a^{-1}}\ \underbrace{b^2(\overset{=1}{aba^{-1}b^{-1}})^{-3}b^{-2}} \in \ker \Phi$$

$$\pi_1(T) = \langle a, b \mid aba^{-1}b^{-1} = 1 \rangle$$

Exercise 11.2: i) $\underbrace{S^1 \vee \dots \vee S^1}_{m} \overset{\text{homeo}}{\cong} \underbrace{S^1 \vee \dots \vee S^1}_{n} \Rightarrow m = n$

ii) " $\underset{\text{hom. equivalent}}{\simeq}$ " " "

i)



$X = \bigvee_m S^1 \quad (m>1) \qquad Y = \bigvee_n S^1$

$X \setminus \{p\} \cong \bigsqcup_m]0,1[$ ← m connected components

$\cong$

$Y \setminus \{f(p)\} \xrightarrow{f(p) \neq q}$ 1 connected component ✗

$\xrightarrow{f(p) = q}$ n connected components

$\Downarrow$

$m = n$

ii) Can't use same argument:

$\mathbb{R} \simeq \mathbb{R}^2$ but $\underset{\text{disconnected}}{\mathbb{R} \setminus \{0\}} \not\simeq \underset{\text{connected}}{\mathbb{R}^2 \setminus \{(0,0)\}}$

We use π_1

$$X \simeq Y \Rightarrow \begin{array}{ccc} \pi_1(X) & \cong & \pi_1(Y) \\ \text{\rotatebox{90}{$\cong$}} & & \text{\rotatebox{90}{$\cong$}} \\ F_m & \overset{?}{\cong} & F_n \end{array}$$

How to show $F_m \cong F_n \Rightarrow m = n$?

Easier: $\mathbb{Z}^m \cong \mathbb{Z}^n \Rightarrow m = n$? Yes.

$$\begin{tikzcd} \mathbb{Z}^m \arrow[r, "\varphi"] & \mathbb{Z}^n \arrow[l, bend left, "\psi"] \end{tikzcd}$$

inverse isomorphisms

$\Rightarrow$ linear maps

$\Rightarrow$ inverse matrices of size $m \times n$, $n \times m$ w/ integer coefficients.

$\Rightarrow m = n$.

Abelianization. G group, let $[G, G]$ be subgroup of G generated by

$$\{\underline{ghg^{-1}h^{-1}} : g, h \in G\} \trianglelefteq G$$

$\uparrow$ normal

Define $G' = G/[G, G] \leftarrow$ abelian.

<u>Fact</u>: Abelianization of F_n is $\mathbb{Z}^n$

$$F_n/[F_n, F_n] \xrightarrow{\cong} \mathbb{Z}^n \quad \text{Surjective}$$

$$[a_i] \longmapsto (0, \dots, 0, 1, 0, \dots, 0)$$

$$\bigvee_m S^1 \simeq \bigvee_n S^1 \Rightarrow \underset{\substack{\cong \\ F_m}}{\pi_1\Big(\bigvee_m S^1\Big)} \cong \underset{\substack{\cong \\ F_n}}{\pi_1\Big(\bigvee_n S^1\Big)}$$

$$\Rightarrow \underset{\substack{\cong \\ \mathbb{Z}^m}}{F_m'} \cong \underset{\substack{\cong \\ \mathbb{Z}^n}}{F_n'} \Rightarrow m = n.$$

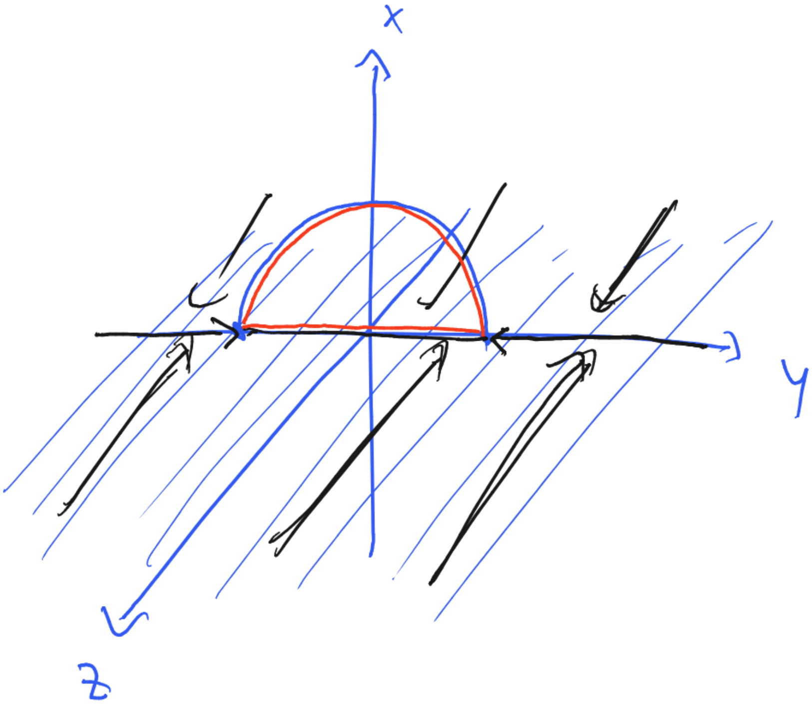
$$\mathbb{Z}^m \cong \mathbb{Z}^n$$

<u>Remark</u>: For every $n > 0$ F_2 has a subgroup isomorphic to F_n.

<u>Remark</u>: $\pi_1(X)' = H_1(X)$.

first homology group.

<u>Exercise 11.3 i)</u>: $X = \{(0, y, z) \in \mathbb{R}^3\} \cup \{(x, y, 0) \in \mathbb{R}^3 : x^2 + y^2 = 1,\ x \geq 0\}$



$$Y = \{(x,y,0) : x^2+y^2=1,\ x\geqslant 0\} \cup \{(0,y,0)\in\mathbb{R}^3\}$$

$$Z = \{ \quad \text{''} \quad \} \cup \{(0,y,0) : |y|\leq 1\}.$$

$$\cong$$

$$S^1$$

① H deformation retract $X \rightsquigarrow Y$

Semi-circle

$$H_t(x,y,0) = (x,y,0)$$

$$H_t(0,y,z) = (0,y,(1-t)z)$$

② $Y \rightsquigarrow Z$ via G

$$G_t(x,y,0) = (x,y,0)$$

$$G_t(0,y,0) = \begin{cases} (0,y,0) & \text{if } |y|\leq 1 \\ \left(0,(1-t)y+t\frac{y}{|y|},0\right) & \text{otherwise} \end{cases}$$

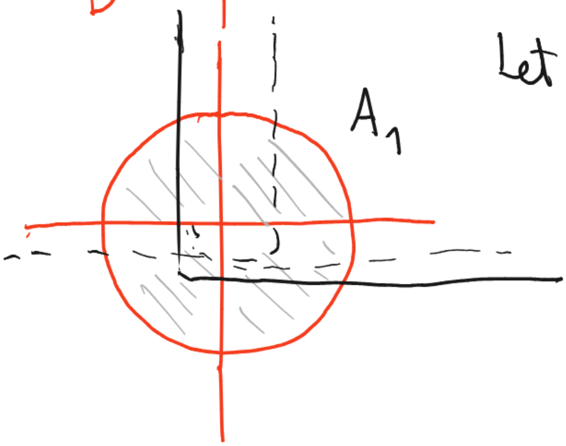
$$\pi_1(X)\cong\pi_1(Y)\cong\pi_1(Z)$$
$$\cong\pi_1(S^1)\cong\mathbb{Z}$$

Exercise 11.6 :

$$X = S^2 \cup \{x=0\} \cup \{y=0\} \cup \{z=0\} \subseteq \mathbb{R}^3$$

Compute $\pi_1(X)$

2D picture



Let $A_1 = X \cap \{(x,y,z) : x,y,z > -\frac{1}{2}\}$

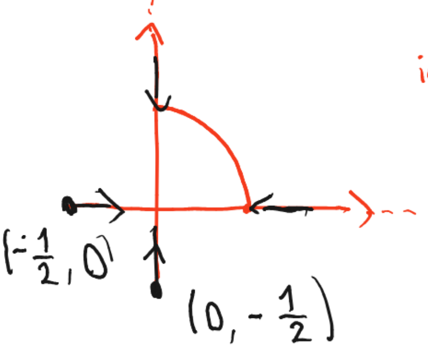
$A_2 = \dots$

$\vdots$

A_8

$X = A_1 \cup \dots \cup A_8$

2D



in 2D (!)

$\simeq S^1$

$\cong S^2$

take linear deformation retract

A_1 deformation retracts to

$\tilde{A}_1 = \{(x,y,z) : x^2+y^2+z^2=1,\ x,y,z \geq 0\}$.

$\cup \{(x,y,z) : x,y,z \geq 0$ and $x=0$ or $y=0$ or $z=0\}$

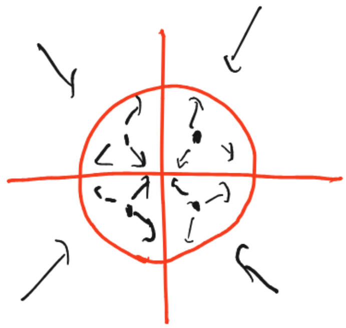
$$\pi_1(A_i) \cong \pi_1(\underset{\substack{\cong \\ S^2}}{\tilde{A}_i}) \cong \pi_1(S^2) \cong \{1\}.$$

$A_i \cap A_j$ is path-connected

$$\overset{\text{SvK}}{\rightsquigarrow} \quad \Phi : \mathop{\ast}_{i=1}^{8} \underset{\substack{\cong \\ \{1\}}}{\pi_1(A_i)} \longrightarrow \pi_1(X) \qquad \text{surjective}$$

$$\Rightarrow \pi_1(X) \cong \{1\}.$$

Another way



Similar to 10.8

$$X \simeq \mathbb{R}^3 \setminus \{8 \text{ points}\}$$

$\rightsquigarrow \pi_1(X) \cong \{1\}$ 10.9

Exercise 11.7: $\gamma_1, \gamma_2 \subseteq S^3 \subseteq \mathbb{R}^3$ linked circles.

$$\pi_1(\mathbb{R}^3 \setminus (\gamma_1 \cup \gamma_2)), \quad \pi_1(S^3 \setminus \gamma_1 \cup \gamma_2)$$

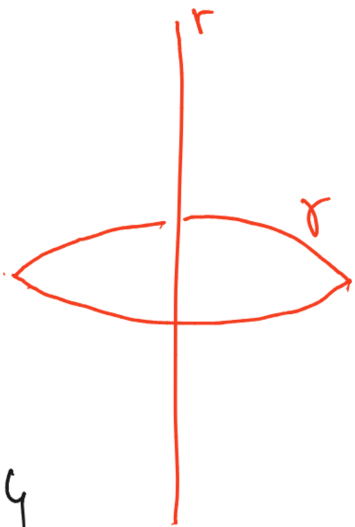
S^3 case

Take $p \in \gamma_1$ and consider stereographic projection

$$S^3 \setminus \{p\} \xrightarrow{\cong} \mathbb{R}^3 \qquad S^3 \setminus (\gamma_1 \cup \gamma_2) \to \mathbb{R}^3 \setminus (r \cup \gamma)$$

circles passing through p $\longmapsto$ line $\rightsquigarrow$

circle not passing through p $\longmapsto$ circle



Wlog $r = \{(0,0,z)\}$

$\gamma = \{(x,y,0) : x^2 + y^2 = 1\}$

Exercise 9.10: $\pi_1(\mathbb{R}^3 \setminus (r \cup \gamma))$

$$\mathbb{R}^3 \setminus r \cong S^1 \times \mathbb{R}^+ \times \mathbb{R}$$ in cylindrical coordinates

$$(R\cos\theta, R\sin\theta, z) \longleftarrow\!\shortmid (\theta, R, z)$$

$\gamma = \{(\theta, R=1, z=0) : \theta \in S^1\}$

$$\mathbb{R}^3 \setminus r \cup \gamma \cong (S^1 \times \mathbb{R}^+ \times \mathbb{R}) \setminus S^1 \times \{(1,0)\}$$

$$\cancel{N} = S^1 \times \underbrace{(\mathbb{R}^+ \times \mathbb{R} \setminus \{(1,0)\})}_{\cong \mathbb{R}^2 \setminus \{(0,0)\}}$$

$$\Rightarrow \pi_1(\mathbb{R}^3 \setminus r \cup \gamma) \cong \pi_1(S^1 \times (\mathbb{R}^2 \setminus \{(0,0)\}))$$

$$\begin{array}{c} \| \wr \\ \pi_1(S^3 \setminus \gamma_1 \cup \gamma_2) \end{array} \qquad \cong \mathbb{Z} \times \mathbb{Z}$$

<u>$\mathbb{R}^3$ case</u>

Take $q \in S^3 \setminus \gamma_1 \cup \gamma_2$.

Stereographic projection induces homeo

$$(S^3 \setminus \gamma_1 \cup \gamma_2) \setminus \{q\} \xrightarrow{\cong} \mathbb{R}^3 \setminus \tilde{\gamma}_1 \cup \tilde{\gamma}_2$$

last time $\rightsquigarrow$ $\pi_1(\mathbb{R}^3 \setminus \tilde{\gamma}_1 \cup \tilde{\gamma}_2) \cong \pi_1((S^3 \setminus \gamma_1 \cup \gamma_2) \setminus \{q\})$

ex 10.1 $\cong \pi_1(S^3 \setminus \gamma_1 \cup \gamma_2)$

$\cong \mathbb{Z} \times \mathbb{Z}$

Exercise 11.8

$$X = \underbrace{\{(z,w)\in\mathbb{C}^2 : |z|=|w|=1\}}_{=\,A} \cup \underbrace{\{(1,w)\in\mathbb{C}^2\}}_{=\,B} \subseteq \mathbb{C}^2$$

$\pi_1(X)$

A a small open set around
B " " "

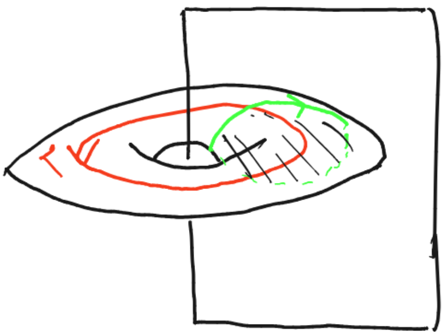
$A \simeq S^1 \times S^1$ $\qquad$ $B \simeq \mathbb{C}$

$\pi_1(A) \cong \underset{\alpha}{\mathbb{Z}} \times \underset{\beta}{\mathbb{Z}}$ $\qquad$ $\pi_1(B) \cong \{0\}$.

Idea: Use SvK with (small open sets around) A and B

$\alpha = [(e^{2\pi i t}, 1)]$. $\qquad$ $\beta = [(1, e^{2\pi i t})]$

$A \cap B = \{(1,w) : |w| = 1\} \cong S^1$



SvK: $\Phi : \underset{\substack{\cong \\ \underset{\alpha}{\mathbb{Z}} \times \underset{\beta}{\mathbb{Z}}}}{\pi_1(A)} * \overset{\{0\}}{\underset{\shortparallel}{}}\pi_1(B) \longrightarrow \pi_1(X)$

Ker Φ generated by image of

$\varphi : \pi_1(A \cap B) \longrightarrow \pi_1(A)$

$\pi_1(A \cap B) \cong \pi_1(S^1) \cong \mathbb{Z}$

$\pi_1(A) \cong \pi_1(S^1 \times S^1) \cong \underset{\alpha}{\mathbb{Z}} \times \underset{\beta}{\mathbb{Z}}$

$\alpha \rightsquigarrow (1,0)$

$1 \longmapsto \beta$

$(0,1)$

$\pi_1(X) \cong \pi_1(A) / \mathrm{Ker}\,\Phi \cong \underset{\alpha}{\mathbb{Z}} \times \cancel{\underset{\beta}{\mathbb{Z}}} / \langle \beta \rangle$

$\cong \mathbb{Z}$