Exercise Class 25/05.

Coverings of the torus.

12.3. Let T^2 be the standard torus. For every subgroup H of $\pi_1(T^2)$, find a covering $g : X \to T^2$ such that $\mathrm{Im}(g_*) = H$.

Solution

Consider $p : \mathbb{R}^2 \to T^2$ the universal covering of $T^2 = S^1 \times S^1$ ($\mathbb{R}^2$ is simply connected)

given by

$p(t,s) = (\cos(2\pi t), \sin(2\pi t), \cos(2\pi s), \sin(2\pi s))$

$\pi_1(T^2) \underset{\text{iso}}{\overset{\varphi}{\cong}} \underbrace{G(\mathbb{R}^2, p)}_{\text{group of deck transformations}}$

Since $\mathbb{R}^2$ is simply connected then indeed $\pi_1(T^2) \underset{\text{iso}}{\cong} G(\mathbb{R}^2, p)$

with $\varphi : \underbrace{\pi_1(T^2) = \mathbb{Z}^2}_{\text{recall}} \to \underbrace{G(\mathbb{R}^2, p)}_{= \{g \,:\, g \text{ isomorphism } g : \mathbb{R}^2 \to \mathbb{R}^2\}}$

given by:

$\underbrace{\varphi((n,m))}_{\in G(\mathbb{R}^2, p)}(t,s) := (t+n, s+m)$ i.e. $\varphi((n,m)) = g : \mathbb{R}^2 \to \mathbb{R}^2$.

$g(t,s) = (t+n, s+m)$

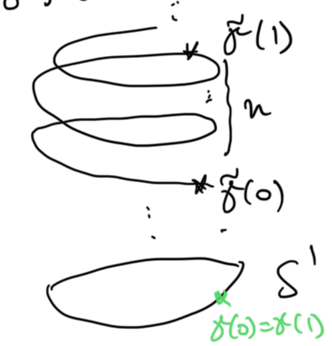
if $\gamma : [0,1] \to T^2$ loop

$\varphi[\gamma](\tilde{\gamma}(0)) = \tilde{\gamma}(1)$ where $\tilde{\gamma}$ is the lift of γ

intuition: think about the circle.

$\varphi_{S^1}(n)(t) = t+n$

$\varphi_{S^1}[\gamma](\tilde{\gamma}(0)) = \tilde{\gamma}(1)$



Observe that $\varphi(H) \triangleleft G(\mathbb{R}^2, p)$, $\varphi(H) \underset{iso}{\cong} H$

$\uparrow$ why? because $\pi_1(T^2) = \mathbb{Z}^2$ is abelian.

$X := \mathbb{R}^2/\varphi(H)$ and define $g : X \to T^2$ given by

$g([x]) = p(x)$.

g is well-defined because if $[x]=[y]$ then $p(x)=p(y)$

Note that for $r : \mathbb{R}^2 \to X$ quotient map then

$g \circ r = p$ $(*)$

Check g is a covering map.

Show $\operatorname{Im} g_*(\pi_1(X)) = H$

take $[\gamma] \neq 0 \in \pi_1(T^2)$, $\gamma : [0,1] \to T^2$ loop.

Consider its lift $\tilde{\gamma} : [0,1] \to T^2$.

Then $r \circ \tilde{\gamma} : [0,1] \to X$ is a loop } recall $X = \mathbb{R}^2/\varphi(H)$

iff $\varphi([\gamma]) \in \varphi(H)$

iff $[\gamma] \in H$. } this is clear

$g_*(\overbrace{[r \circ \tilde{\gamma}]}^{\text{loop in } X}) = [g \circ r \circ \tilde{\gamma}] \underset{(*)}{=} [p \circ \tilde{\gamma}] \underset{\tilde{\gamma} \text{ lift}}{=} [\gamma] \in H$.

then $\operatorname{Im}(g_*) = H$.

12.4. Determine two covering maps $p: X \to T^2$, $p': X \to T^2$

such that:

- $p: X \to T^2$, $p': X \to T^2$ have the same number of sheets
- $\nexists$ homeomorphism $\phi: X \to X'$, $\psi: T^2 \to T^2$ $p' \circ \phi = \psi \circ p$.

$$\begin{tikzcd} X \arrow[r, "p"] \arrow[d, "\phi"'] & T^2 \arrow[d, "\psi"] \\ X' \arrow[r, "p'"'] & T^2 \end{tikzcd}$$

<u>Solution</u> identify T^2 by $\mathbb{R}^2/\mathbb{Z}^2$ $\quad (X = \mathbb{R}^2/\mathbb{Z}^2)$

define

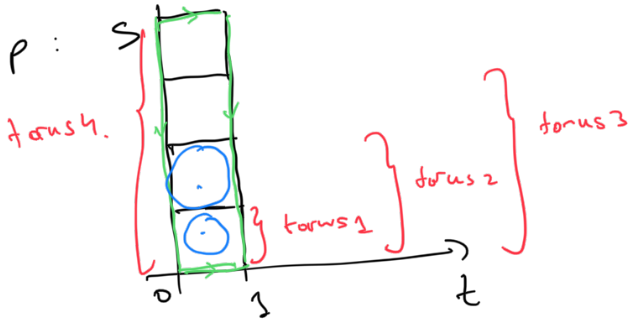
$p([(t,s)]) = [(t,4s)]$

$p'([(t,s)]) = [(2t,2s)]$, $p'_*: \pi_1(T^2) \to \pi_1(T^2)$

$\mathbb{Z}^2 \to 2\mathbb{Z}\times 2$

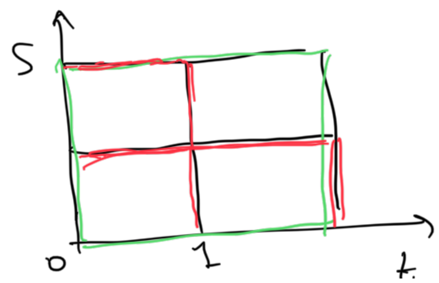
$\forall\ [(t,s)] \in \mathbb{R}^2/\mathbb{Z}^2$

<u>Picture</u> p :



$p^{-1}([t,4s]) = \{[t,s], [t,2s], [t,3s], [t,4s]\}$

p'



$p'^{-1}([2t,2s]) = \{[t,s], [2t,s], [t,2s], [2t,2s]\}$

then we conclude that p, p' have the same number of sheets, p, p' have degree 4.

Assume $\exists$ homeomorphisms $\phi : X \to X'$, $\psi : T^2 \to T^2$ such that $p' \circ \phi = \psi \circ p$.

$$
\begin{aligned}
2\mathbb{Z} \times 2\mathbb{Z} &= p'_*(\mathbb{Z} \times \mathbb{Z}) \\
\phi_* \text{ iso.} \quad &= p'_* \circ \phi_* (\pi_1(T^2)) \\
&= \psi_* \circ p_* (\pi_1(T^2)) \\
\text{def of } p, p_* \quad &= \psi_*(\mathbb{Z} \times 4\mathbb{Z})
\end{aligned}
$$

then $\psi_* : \mathbb{Z}^2 \to \mathbb{Z}^2$ is an isomorphism such that

$$\psi_*(\mathbb{Z} \times 4\mathbb{Z}) = 2\mathbb{Z} \times 2\mathbb{Z} \qquad (**)$$

$$\mathbb{Z} \times \mathbb{Z} / \mathbb{Z} \times 4\mathbb{Z} = \mathbb{Z}/4\mathbb{Z}$$

compute

$$\mathbb{Z} \times \mathbb{Z} / 2\mathbb{Z} \times 2\mathbb{Z} = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$$

compute.

by $(**)$ $\Rightarrow$ $\mathbb{Z}/4\mathbb{Z} \simeq \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$

false.

Hence, contradiction.

Borsuk Ulam theorem (ex 7) & applications (ex 8)

12.7 (i) Prove that $\forall f : S^2 \to \mathbb{R}$ $\exists$ $x \in S^2$ such that $f(x) = f(-x)$

(ii) Prove that $\forall f: S^2 \to \mathbb{R}^2$ $\exists x \in S^2$ such that

$f(x) = f(-x)$.

Proof

(i) $h: S^2 \to \mathbb{R}$, $h(x) = f(x) - f(-x)$

$h(S^2)$ is connected since h is cts

and S^2 is connected.

because $h(x) = -h(-x)$ and $h(S^2)$ is connected

then $\exists x_0$ such that $h(x_0) = 0$.

(ii) assume $\forall x$ $f(x) \neq f(-x)$

$g: S^2 \to S^1$

$$g(x) = \frac{f(x) - f(-x)}{\| f(x) - f(-x) \|}$$

g is well defined, continuous

Define $\gamma : [0,1] \to S^2$

$\gamma(s) = (\cos 2\pi s , \sin 2\pi s, 0)$

equator of S^2.

$h: [0,1] \to S^1$ given by $h = g \circ \gamma$

and $\exists \tilde{h} : [0,1] \to \mathbb{R}$ with respect to

the universal cover $p: \mathbb{R} \to S^1$

$p(t) = (\cos 2\pi t, \sin 2\pi t)$

because $g(x) = -g(-x)$ we have:

$\ldots \sin(2\pi s + \pi))$

$$h(s+\tfrac{1}{2}) = g((\cos(2\pi s+\pi), \sin(2\pi s \dots$$

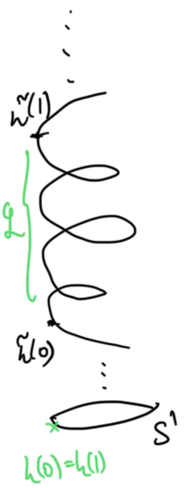
$$= -g((\cos(2\pi s), \sin(2\pi s)))$$

$$= -h(s).$$

$\forall\, s \in [0, \tfrac{1}{2}]$

then $\tilde{h}(s+\tfrac{1}{2}) = \tilde{h}(s) + \frac{q(s)}{2}$, $q(s)$ odd number depending on s

$$\left\{\begin{array}{l} \overset{\in\mathbb{R}}{t_1} + \dfrac{\overset{\text{odd}}{q}}{2} = t_2 \quad , \quad \text{compute } p(t_2) \text{ with respect to } p(t_1) \\[2ex] p(t_2) = p(\cos(2\pi t_1 + \pi q), \sin(2\pi t_1 + \pi q)) \\[1ex] \qquad\ = -p(t_1) \end{array}\right.$$



Note that $\tilde{h}$ is continuous $\Rightarrow q(s) = q$ a constant.

Moreover $\tilde{h}(1) = \tilde{h}(\tfrac{1}{2}) + \frac{q}{2} = \tilde{h}(0) + q$.

q is the number of times h turns around S^1.

$[h] = g_*[\gamma] \in \pi_1(S^1)$

$\Rightarrow g_*[\gamma] \neq 0$ because q is odd

However $[\gamma] = 0 \in \pi_1(S^2)$ (since $\pi_1(S^2) = \{0\}$)

Thus we have a contradiction.

12.8. $A_1 \dots A_k$ closed sets , $\bigcup_{i=1}^{k} A_i = S^2$

Show that for $k \leq 3$ $\exists\, i \in \{1 \dots k\}$ such that

A_i contains a pair of antipodal points.

How about $k \geqslant 4$.

Solution

$k=3$ Assume by contradiction $\exists A_1, A_2, A_3 \subseteq S^2$

$A_1 \cup A_2 \cup A_3 = S^2$

and no A_i contains a pair of antipodal points.

Define $B_i = -A_i = \{-x \in S^2 : x \in A_i\}$

Consider $f_i : S^2 \to \mathbb{R}$: $f_i(x) = d(x, B_i) - d(x, A_i)$

where d is the distance on S^2.

$\forall x \in A_i$ $f_i(x) = d(x, B_i) > 0 > -d(-x, A_i)$

$= f_i(-x)$

$(A_i \cap B_i = \phi)$

$\Rightarrow f_i(x) \neq f_i(-x)$. $\underline{\forall x \in A_i}$ or $-x \in A_i$

$(\square)$

Define $f : S^2 \to \mathbb{R}^2$

$f(x) = (f_1(x), f_2(x))$, f continuous

by 12.7 (ii) $\Rightarrow \exists x_0$ such that $f(x_0) = f(-x_0)$

$\Rightarrow f_1(x_0) = f_1(-x_0)$ and $f_2(x_0) = f_2(-x_0)$

then $x_0, -x_0 \notin A_1 \cup A_2$ (by $(\square)$)

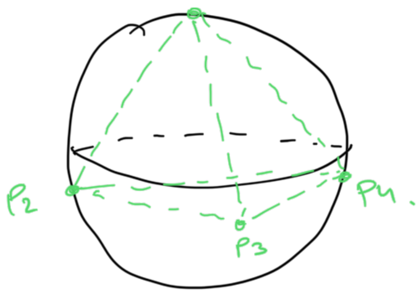
Thus $x_0, -x_0 \in A_3$ because $S^2 = A_1 \cup A_2 \cup A_3$:

We have a contradiction.

$k=4$ the statement is false.

$(k = 5, 6 \ldots)$

follow,



Hatcher: project the four faces of the tetrahedron radially on S^2.

Hence obtain the four sets A_i.

or p_1, p_2, p_3, p_4.

$$d(p_i, p_j) = c < \pi$$

we define $A_i = \overline{B(p_i, c)}$

$$= \{x \in S^2 : d(x, p_i) \le c\}.$$