Topology - Lecture 9

<u>Thm</u> (Heine-Borel) For a subset $K \subset \mathbb{R}^n$ the following two statements are equivalent:

① K is <u>compact</u> (open covers...)

② K is <u>closed</u> and <u>bounded</u>.

[cf. w/ Exercise 4.9 for metric spaces]

<u>Pf.</u> ① $\Rightarrow$ ② Assume K compact. Then

- K is bounded: take $O_n := B(0, n)$ $n \in \mathbb{N}_*$
  $\{O_n\}_{n \in \mathbb{N}_*}$ covers $\mathbb{R}^n$, thus K, so $\exists$ <u>finite</u> subcover i.e. $\exists\ \bar{n} \in \mathbb{N}_*$ w/ $K \subset B(0, \bar{n})$
  $\Rightarrow K$ bounded.
- K is closed: by contradiction let $K \subsetneq \overline{K}$
  so pick $x \in \overline{K} \setminus K$. Take $\underbrace{O_n}_{\text{open}} := \mathbb{R}^n \setminus \overline{B(x, 1/n)}$
  $\bigcup O_n = \mathbb{R}^n \setminus \{x\} \supset K$
  but no finite subcover can exist because
  $x \in \overline{K} \Rightarrow \forall\ n \in \mathbb{N}_*\ \exists\ \underbrace{x_n}_{\in K} \in B(x, 1/n)$ $\square$

② $\Rightarrow$ ① Let K be closed and bounded.

Strategy: • a closed interval $[a, b]$ is compact

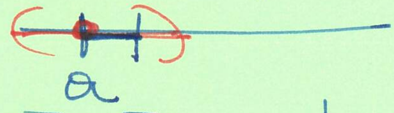
- K bounded $\Rightarrow \exists\ r \gg 1$ w/

$$\underbrace{K}_{\text{closed}} \subset \underbrace{[-r, r] \times \dots \times [-r, r]}_{\text{compact (Tychonoff)}}$$

$\Rightarrow$ K compact $\square$

Let's check that a closed interval $[a,b]$ is compact.

$a<b$ (else nothing to prove!)

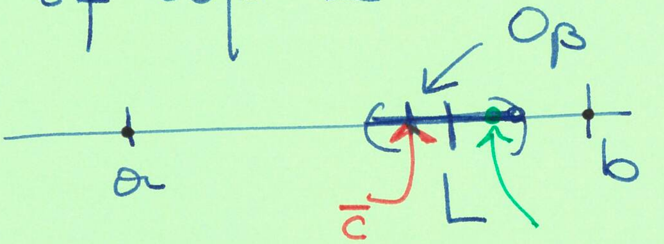


$\{O_\alpha\}_\alpha$ cover of $[a,b] \Rightarrow \exists\, \bar{\alpha}$ s.t. $a \in O_{\bar{\alpha}}$, but $O_{\bar{\alpha}}$ open $\Rightarrow \exists\, c>a$ w/ $[a,c] \subset O_{\bar{\alpha}}$. Then define

$$L := \sup\{c : [a,b] \text{ w/ } [a,c] \subset \text{finitely many } \{O_\alpha\}\}$$

well-defined, $L>a$, $L \leq b$.

by def of sup $\exists\, \bar{c}>a$ w/
- $\bar{c}<L$
- $\bar{c} \in O_\beta$

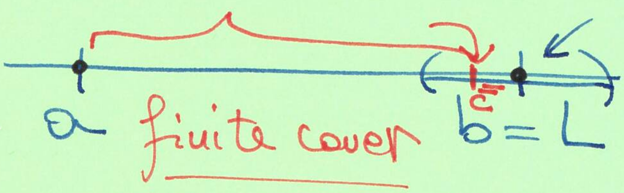


- $[a,\bar{c}]$ covered by finitely many $\{O_\alpha\}$

hence $[a,\bar{c}] \cup O_\beta$ trivially covered by finitely many sets of $\{O_\alpha\}$

This leads to a contradiction unless $L=b$.

So the picture is

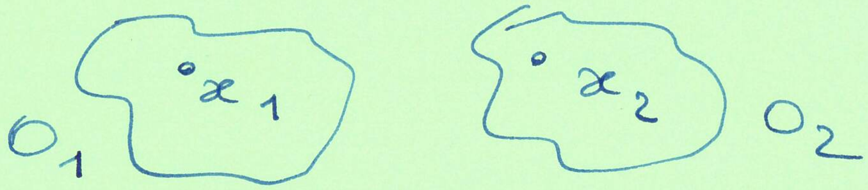


Same argument as above $\Rightarrow [a,b]$ has a finite subcover. $\square$

Separation Axioms

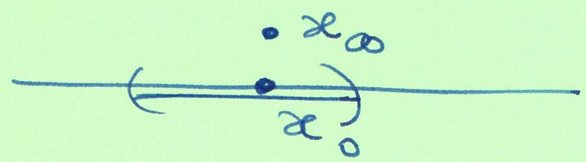
<u>Def.</u> A top. space X is called <u>Hausdorff</u> (or T_2) if $\forall x_1 \neq x_2 \in X \ \exists \begin{cases} O_1 \ni x_1 \\ O_2 \ni x_2 \end{cases}$ open neighborhoods w/ $O_1 \cap O_2 = \emptyset$.

<u>Picture</u>



<u>Examples:</u>

1. any metric space (X, d) is Hausdorff.
   <u>why?</u> $x_1 \neq x_2 \Longrightarrow d(x_1, x_2) = \delta > 0$
   pick $O_1 := B(x_1, \delta/3)$ $O_2 := B(x_2, \delta/3)$
   triangle inequality $\Longrightarrow O_1 \cap O_2 = \emptyset$.
2. X set, w/ trivial topology is <u>not</u> H. if $\text{card}(X) > 1$.
3. $\mathbb{R}$ w/ <u>cofinite topology</u>. Any two (non-empty) open sets have cofinite intersection hence must intersect.
4. Line w/ double point



Trivialities: let X be Hausdorff. Then:

a) points are closed

b) any subspace $Y \subset X$ is Hausdorff

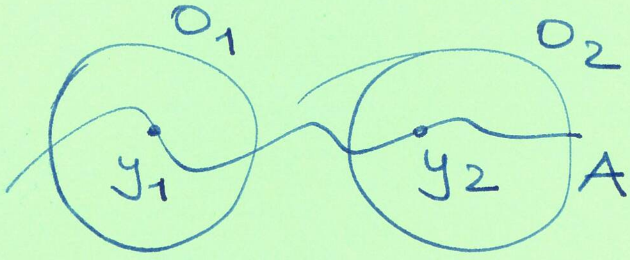
c) if X_1, X_2 are Hausdorff then so $X_1 \times X_2$
(q.: how about viceversa?)

Pf. a)

$$X \setminus \{x\} = \underbrace{\bigcup_{y \neq x} V_y}_{\text{open}}$$

$\to$ so $X \setminus \{x\}$ open hence $\{x\}$ closed.

b)

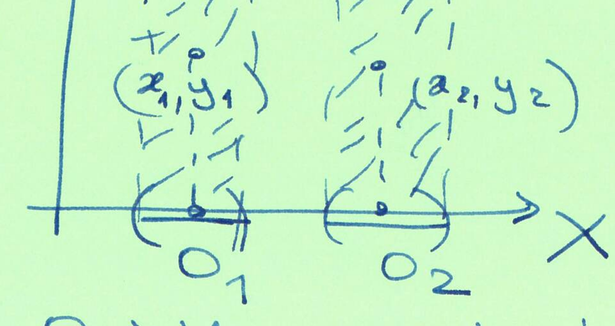


$O_1 \cap A$, $O_2 \cap A$ open in subspace topology and disjoint.

c) $(x_1, y_1) \neq (x_2, y_2)$

$\Longrightarrow$ either $x_1 \neq x_2$ or $y_1 \neq y_2$

First case



$O_1 \ni x_1$

$O_2 \ni x_2$

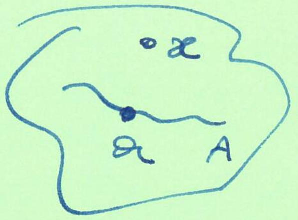
$O_1 \cap O_2 = \emptyset$ in X

So take $O_1 \times Y$, $O_2 \times Y$ in $X \times Y$.

Two key facts:

Prop. 1: A compact subspace of a Hausdorff space is closed. ($\sim$ partial Heine-Borel)

Pf. X be Hausdorff, $A \subset X$ compact



X $\quad x \in X \setminus A$

Claim: $\exists$ disjoint open sets U, V w/ $U \ni x$, $A \subset V$.

$a \in A \xrightarrow[\text{Hausdorff}]{\text{invoke}} U_x^{(a)}, \; V_a$ $\quad \{V_a\}$ covers A

$\swarrow$ A cpt.

finite subcover $V_1, \dots, V_n$

take $V = V_1 \cup \dots \cup V_n$

$U = U_x^{(1)} \cap \dots \cap U_x^{(n)}$ $\quad \square$

Prop. 2 (homeo criterion) $f : X \to Y$ bijective, continuous. If X compact and Y Hausdorff then f is a homeomorphism.

Pf f^{-1} continuous iff f closed i.e. $\forall C \subset X$ closed, have $f(C) \subset Y$ closed.

But $C \subset X$ closed $\Rightarrow$ C compact $\Rightarrow$ $f(C) \subset Y$ compact $\xrightarrow{\text{Prop 1}}$ $f(C)$ closed $\quad \square$