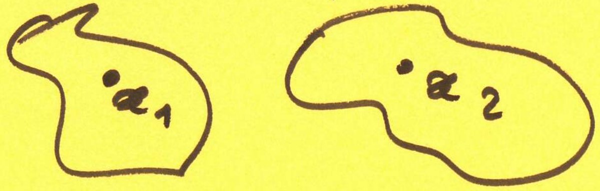
Topology - Lecture 10

Separation Axioms

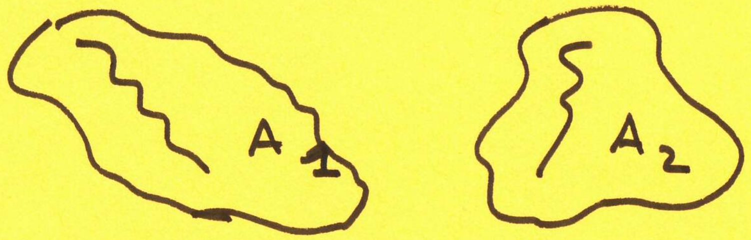
last time: X is Hausdorff (or T_2)

if



this time: X is **normal** (or T_4)

if it is T_2, plus



$\forall A_1, A_2$ closed $A_1 \cap A_2 = \phi$

$\exists \begin{cases} U_1 \supset A_1 \\ U_2 \supset A_2 \end{cases}$ open sets w/ $U_1 \cap U_2 = \phi$

rmks • if X is T_2 then points are **closed**
(i.e. T_4 is strictly stronger than T_2)

- have a look at "Normal Space" on Wikipedia to see the whole hierarchy of separation axioms.

First Fact:

$$\left\{ \begin{matrix} \text{Hausdorff} \\ \text{top. spaces} \end{matrix} \right\} \supsetneq \left\{ \begin{matrix} \text{normal} \\ \text{top. spaces} \end{matrix} \right\}$$

Example : a space which is Hausdorff but not normal.

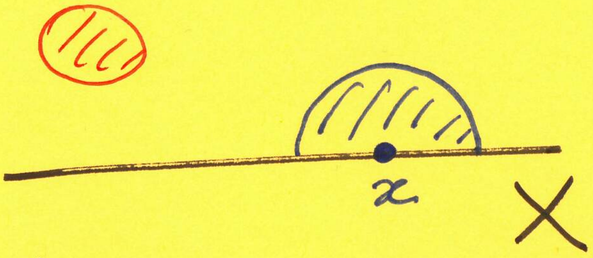
As a set, consider $X := \{(x,y) \in \mathbb{R}^2 : y \geq 0\}$

set $X' := \{(x,y) \in \mathbb{R}^2 : y > 0\}$.
We let τ be the topology on X generated by the **basis** consisting of the following 2 classes:

a) open balls $B(x,r) \subset X'$ (**Euclidean**)

b) for $x \in X \setminus X'$ take

$$B'(x,r) = \{x\} \cup (X' \cap B(x,r))$$



check (criterion pg. 8) this **does** define a top.

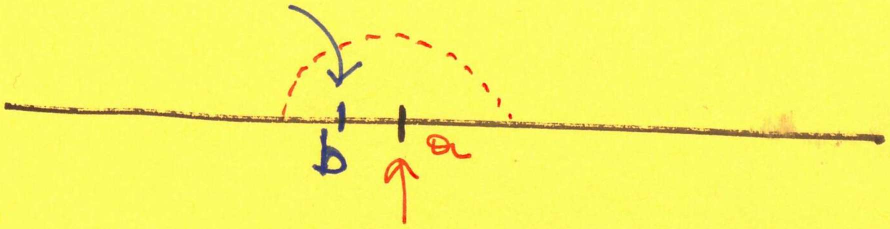
(X, τ) is Hausdorff because, given any $x_1, x_2 \in X$, $x_1 \neq x_2$ are contained in **disjoint** elements of the basis above (reality check).

not normal why? $L := X \setminus X'$

- any subset $Y \subset L$ is closed in X (check that $X \setminus Y$ is open ...)
- $L = A \cup B$

$\xleftarrow{\text{rationals}}$ A; B $\xleftarrow{\text{irrationals}}$

given $a \in A$ if $\overset{U}{A}$, $\overset{V}{B}$ could be separated by open sets then $\exists\, r = r(a)$ w/ $B'(a,r) \subset U$



$\forall\, r' > 0$ the sets

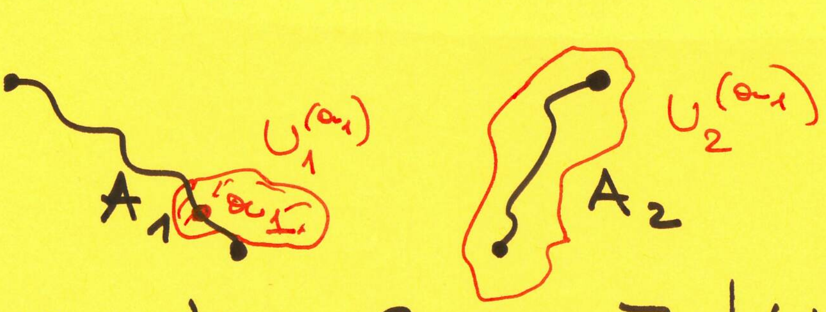
$B'(a,r)$, $B'(b,r')$ are not disjoint which maks the condition $U \cap V = \phi$ <u>impossible</u> ($\Rightarrow$ contradiction)

Conclusion: (X, τ) is not normal.

~ ∘ ~

<u>Prop. 1</u>: A <u>compact</u> Hausdorff top. space is normal.

<u>Pf.</u> let X be our compact H. space take $A_1, A_2 \subset X$ closed, disjoint.



last time: $\forall\, a_1 \in A_1 \ \exists \begin{cases} U_1^{(a_1)} \ni a_1 \\ U_2^{(a_1)} \supset A_2 \end{cases}$ w/ $U_1^{(a_1)} \cap U_2^{(a_1)} = \phi$.

Now, play this same game as we vary $a_1 \in A_1$

The family $\{U_1^{(\alpha_1)}\}_{\alpha_1 \in A_1}$ covers A_1

so (by compactness) $\exists$ finite subcover of A_1

$$U_1^1, \dots, U_1^k$$

$$U_2^1, \dots, U_2^k \longleftarrow$$ corresponding open sets containing A_2

Set

$U_1 := U_1^1 \cup \dots \cup U_1^k$ open set covering A_1

$U_2 := U_2^1 \cap \dots\dots \cap U_2^k$ open set covering A_2

by construction $U_1 \cap U_2 = \phi$, i.e.

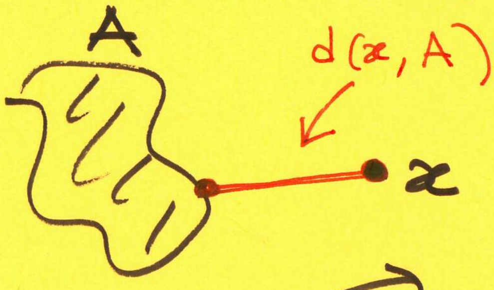
we conclude that X is normal. □

Prop. 2: Metric spaces are normal.

[$(X, d) \longrightarrow$ topology induced by this metric

we know such top. space is T_2

we check now it is T_4]



$A \subset X$

$$d(x, A) := \inf_{a \in A} d(x, a)$$

closest point distance

Facts: 1) $f(x) := d(x, A)$

$f : X \longrightarrow \mathbb{R}$ **continuous**

2) $f(x) = 0 \iff x \in \overline{A}$

Pf. take (X, d) metric space, let $A_1, A_2 \subset X$ be closed and disjoint.

For $i = 1, 2$ set $f_i(x) := d(x, A_i)$

$$U_1 := \{ x \in X : f_1(x) < f_2(x) \}$$

$$U_2 := \{ x \in X : f_2(x) < f_1(x) \}$$

Claim I : U_1, U_2 are **open** and **disjoint**.

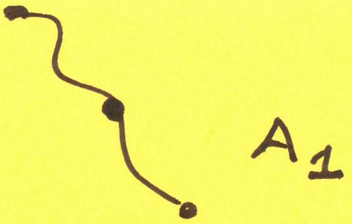
$$U_1 = (f_1 - f_2)^{-1}\big[(-\infty, 0)\big]$$

$$U_2 = (f_1 - f_2)^{-1}\big[(0, +\infty)\big]$$

Claim II: $A_i \subset U_i \quad i = 1, 2$ why?

Let's check e.g. that $A_1 \subset U_1$

$f_1 |_{A_1} = 0$



$f_2 |_{A_1} > 0$

because A_2 is closed : $x \in A_1 \Rightarrow$

$x \in \underbrace{X \setminus A_2}_{\text{open}} \Rightarrow \exists\, \epsilon > 0$ w/ $B(x, \epsilon) \subset X \setminus A_2$

$\Rightarrow f_2(x) \geqslant \epsilon > 0$

So $x \in A_1 \Rightarrow x \in U_1$.

Same story gives $A_2 \subset U_2$. □

Lebesgue number : define a size for an open cover

quantitative scale

Setup: (X, d) metric space, induced top.

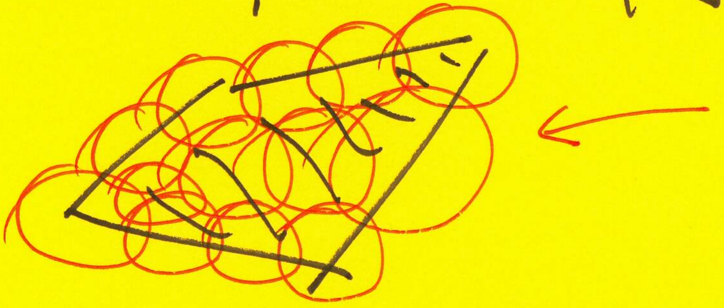
given an open cover of X, say $\{A_\alpha\}$

$\epsilon > 0$ is a Lebesgue number for this cover if

$$\forall x \in X \quad \exists\, \alpha = \alpha(x) \text{ w/ } B(x, \epsilon) \subset A_\alpha .$$

(cf. visual problem at page 38)

(cf.



)

Prop. Every open cover of a compact metric space has a Lebesgue number.

Pf. Let $\{U_\alpha\}$ be a cover of X (w/ (X, d) is our metric space)

X compact $\Longrightarrow \exists$ finite subcover of $\{U_\alpha\}$
$U_1, \dots, U_k$

It is enough to prove that $\exists\, \epsilon > 0$ w/

$\forall x \in X \quad \exists i \in \{1, \dots, k\}$ w/

$$\underbrace{B(x, \epsilon) \subset U_i}$$

$$\Updownarrow$$

rephrase: $d(x, X \setminus U_i) \geq \epsilon$

(if $U_i = X$ for some i, nothing to prove since any $\epsilon > 0$ works)

goal: $f(x) := \max\{d_1(x), \dots, d_k(x)\} \geq \epsilon$

$d_i(x) = d(x, X \setminus U_i)$ (positive, TBD)

$f : X \longrightarrow \mathbb{R}$ continuous

$\{U_1, \dots, U_k\}$ being a cover of X means $\forall x \in X \quad \exists i \in \{1, \dots, k\}$ w/ $x \in U_i$ hence $f(x) \geq d_i(x) > 0$ (*)

by Weierstrass $f : X \to \mathbb{R}$ has a minimum $x_0 \in X$ w/ $\lambda : \underbrace{f(x_0)}_{>0} = \min_{x \in X} f(x)$

$\lambda > 0$ by (*), hence take $\epsilon = \lambda$ ← is Lebesgue n. for the cover

$\square$

Application : uniform continuity

Def. : X, Y metric spaces, $f : X \to Y$ is said **uniformly continuous** if $\forall \epsilon > 0$ $\exists \delta > 0$ such that

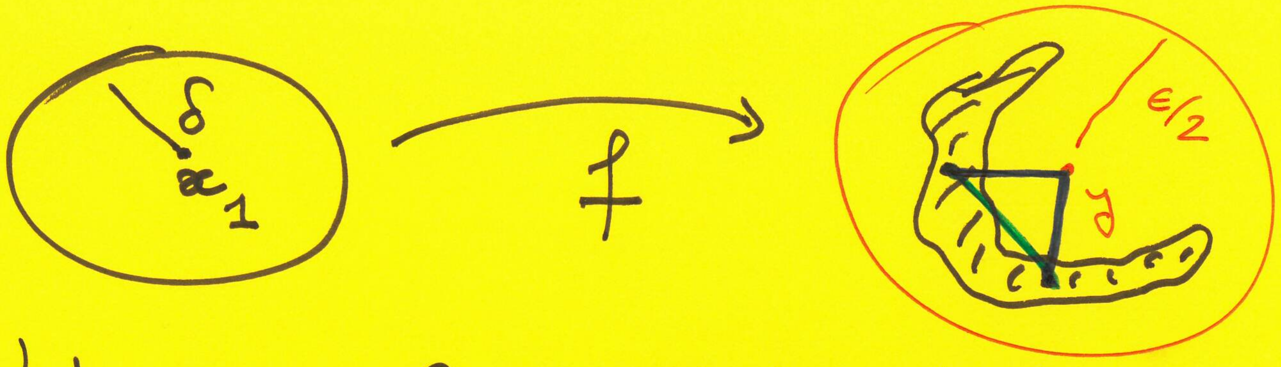
does not depend on the point

$$d_X(x_1, x_2) < \delta \Rightarrow d_Y(f(x_1), f(x_2)) < \epsilon.$$

Recall Heine-Cantor thm.

Prop. X **compact** metric space
Y metric space
Any $f : X \to Y$ continuous is **uniformly** cont.

Pf. given $\epsilon > 0$, consider the open cover of X given by the pre-images $f^{-1}(B(y, \epsilon/2))$ as we vary $y \in Y$.
Let $\delta > 0$ be a Lebesgue number of X w.r.t. this open cover. Then $\forall x \in X$ $B(x, \delta) \subset f^{-1}(B(y, \epsilon/2))$ for some y.
$f(B(x, \delta)) \subset B(y, \epsilon/2)$, hence conclusion comes by triangle inequality:



Let $x_1, x_2 \in X$ satisfy $d_X(x_1, x_2) < \delta$

So $x_2 \in B(x_1, \delta)$, previous story

$$\begin{aligned} d_Y(f(x_1), f(x_2)) &\leq d_Y(f(x_1), y) + \\ &\quad d_Y(y, f(x_2)) \\ &< \epsilon/2 + \epsilon/2 = \epsilon \end{aligned}$$

□