Infinite Products

$I = \mathbb{N}$, family $\{X_i\}_{i \in I}$ where each X_i is a topological space.

Goal: want to define a "good" topology on the set

$$\prod_{i \in I} X_i = \{ (x_0, x_1, x_2, \dots) : \underbrace{x_i \in X_i} \}$$

set of sequences

1st attempt: mimic what we did for finite product. Consider the topology $\mathcal{O}_{box}$ generated by the basis

$$U_0 \times U_1 \times U_2 \times \dots \times U_i \times \dots$$

where $U_i \subset X_i$ open. This definition is well-posed (cf. criterion pg. 8) but this topology has some issues.

Morally speaking $\mathcal{O}_{box}$ has too many open sets.

Two specific problems:

i) lots of simple maps

$$Z \xrightarrow{\varphi} \prod_{i \in I} X_i$$

that turn out not to be continuous.

E.g. take $Z = X_i = \mathbb{R}$ ← w/ Euclidean top.

$$x \overset{\varphi}{\longmapsto} (x, x, x, \dots)$$

$$U = \left(-\tfrac{1}{2}, \tfrac{1}{2}\right) \times \left(-\tfrac{1}{3}, \tfrac{1}{3}\right) \times \left(-\tfrac{1}{4}, \tfrac{1}{4}\right) \times \dots$$

open set of $\mathcal{O}_{box}$ $\varphi^{-1}(U) = \{0\}$

not open in $\mathbb{R}$

$\Rightarrow$ this "diagonal map" is not continuous.

ii) Tychonoff would be false w.r.t. box topology for infinite products.

Take $X_i = X$ $\forall i \in I$

finite set, $|X| \geq 2$ w/ discrete topology

$\prod_{i \in I} X_i$ w/ box topology is an infinite set w/ discrete top. $\Rightarrow$ not compact.

2nd attempt: consider on $\prod_{i \in I} X_i$

$\mathcal{O}_{prod}$ the topology generated by the basis of open rectangles as above BUT such that $U_i = X_i$ except for finitely many indices $i \in I$. Example:

$$U_0 \times U_1 \times X_2 \times U_3 \times U_4 \times X_5 \times X_6 \times \dots$$

check: these sets (called: cylindrical sets) form a basis for a topology τ_{prod}.

Good properties:

i) $f: Z \longrightarrow \prod_{i \in I} X_i$

have $f = (f_0, f_1, f_2, \dots)$

f continuous $\iff$ each $f_i : Z \longrightarrow X_i$ is continuous.

Pf. $\boxed{\Rightarrow}$ each projection map

$\pi_j : \prod_{i \in I} X_i \longrightarrow X_j$

is continuous, and $f_j = \pi_j \circ f$

$\boxed{\Leftarrow}$ $f^{-1}(U_0 \times U_1 \times U_2 \times \dots) =$

$= f_0^{-1}(U_0) \cap \underbrace{f_1^{-1}(U_1)}_{\text{open in } Z} \cap \dots\dots .$ ← trivial i.e. $= Z$

except for finitely many terms

Finite intersection of open sets $\Rightarrow$ is open $\square$

ii) Tychonoff thm. is true, i.e.

$\prod_{i \in I} X_i$ is compact iff each X_i compact

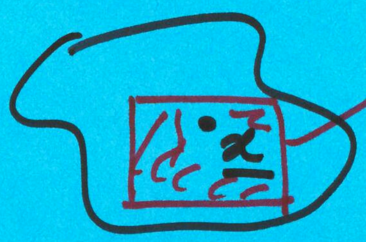
(advice: review proof for $X_1 \times X_2$).

Pf.

1) argue by contradiction: let $\mathcal{O}$ be an open cover of $X = \prod_{i \in I} X_i$ w/ no finite subcover.

2) strategy: produce a "special" point $\underline{x} = (x_0, x_1, x_2, \ldots) \in \prod_{i \in I} X_i$ s.t. any element of the basis (of cylindrical sets) containing $\underline{x}$ cannot possibly be covered by finitely many elements of $\mathcal{O}$.

If so, I'm OK: by covering property $\exists\, O \in \mathcal{O}$ open set covering $\underline{x}$.



this RED set cannot be covered by an element of $\mathcal{O}$

3) select x_0: claim that $\exists\, x_0 \in X_0$ s.t. every basis element of the form $U_0 \times X_1 \times X_2 \times \ldots$

open set in X_0, containing x_0

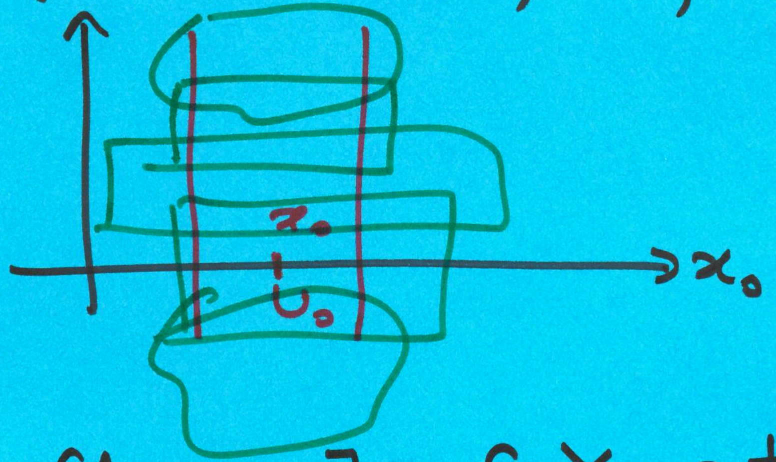
cannot be covered by finitely many elements of $\mathcal{O}$. If not, $\forall x_0 \in X_0$ find $U_0^{(x_0)} \times X_1 \times \ldots$

w/ finite cover via elements of $\mathcal{O}$.

Then $\{U_0^{(x_0)}\}_{x_0 \in X_0}$ open cover of X_0

$\Longrightarrow \exists$ finite subcover $U_0^{x_0^1}, \dots, U_0^{x_0^k}$

of X_0



4) select x_1. Claim: $\exists x_1 \in X_1$ s.t. every basis set of the form

$$\underbrace{U_0 \times U_1}\times X_2 \times X_3 \dots\dots$$

contains (x_0, x_1) is not covered by finitely many elements of $\mathcal{O}$.

If not: $\forall x_1 \in X_1$ there is a well-covered set of the form $U_0^{(x_1)} \times U_1^{(x_1)} \times X_2 \dots\dots$

Then consider $\{U_1^{(x_1)}\}$ cover of X_1

extract a finite subcover

$$U_1^{x_1^1} \quad \dots \quad U_1^{x_1^e}$$

$$U_0^{x_1^1} \quad \dots \quad U_0^{x_1^e}$$

Take intersection $U_0^{x_1^1} \cap \dots U_0^{x_1^e} = \tilde{U}_0$

the strip $\tilde{U}_0 \times X_1 \times X_2 \times \dots$

would have a finite cover via elements of $\mathcal{O}$, but $\tilde{U}_0 \ni x_0$ ← violates the way x_0 was chosen!

open set

5) Now, keep going one coordinate at a time

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Same story for arbitrary products (e.g. $I = \mathbb{R}, \dots$) modulo technical complications. One needs <u>well-ordering</u> principle.