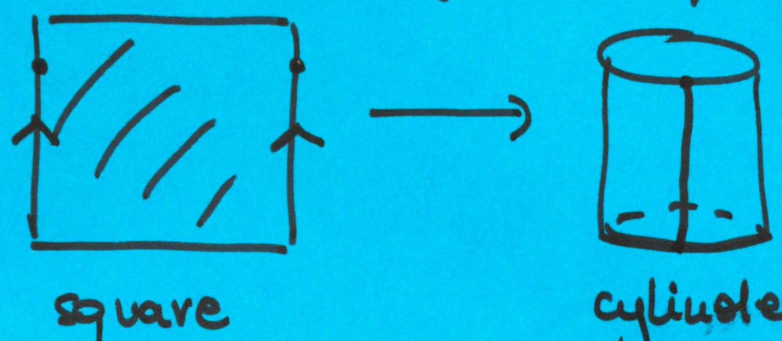
Topology - Lecture 12

Quotient Spaces

motivation: lots of interesting / important spaces are obtained from simpler ones via "gluing" operations

example:



goal: formalize / generalize operation above.

Setup: X top. space, $\sim$ be an equivalence relation on X

at the level of sets we have a well-defined quotient set $X/\sim$

also there is a projection map

$$p : X \longrightarrow X/\sim$$
$$x \longmapsto [x].$$

q. what topology on $X/\sim$?

Let's study the "toy model case" above.

Strategy: describe the topology on the cylinder in purely intrinsic terms.

Prop. Let $h: X \longrightarrow Y$ be a continuous surjective map w/ X compact, Y Hausdorff. Then it is always true that $U \subset Y$ is open if and only if $h^{-1}(U) \subset X$ is open.

Pf. continuity of h gives that

$$U \subset Y \text{ open} \Longrightarrow h^{-1}(U) \text{ open}$$

Converse: copy the proof about homeomorphism criterion. $\square$

Def. the quotient topology on $X/\sim$ is ~~defining~~ defined by declaring

$$U \subset X/\sim \text{ is open} \Longleftrightarrow p^{-1}(U) \subset X \text{ open.}$$

Check: 1) this definition is well-posed

$$p^{-1}(\phi) = \phi, \quad p^{-1}(X/\sim) = X$$

$$p^{-1}\Big(\bigcup_\alpha O_\alpha\Big) = \bigcup_\alpha p^{-1}(O_\alpha)$$

$$p^{-1}\Big(\bigcap_\alpha O_\alpha\Big) = \bigcap_\alpha p^{-1}(O_\alpha)$$

2) with this topology, $p: X \longrightarrow X/\sim$ is continuous (tautology)

3) this topology on $X/\sim$ is indeed the finest topology on $X/\sim$ such that $p: X \to X/\sim$ is continuous.

Examples:

(a) $X = [0,1]^2 \subset \mathbb{R}^2$ w/ Euclidean top.

equivalence relation: $(0,t) \sim (1,t) \quad t \in [0,1]$

(b) $X = [0,1]^2 \subset \mathbb{R}^2$ w/ Euclidean top.

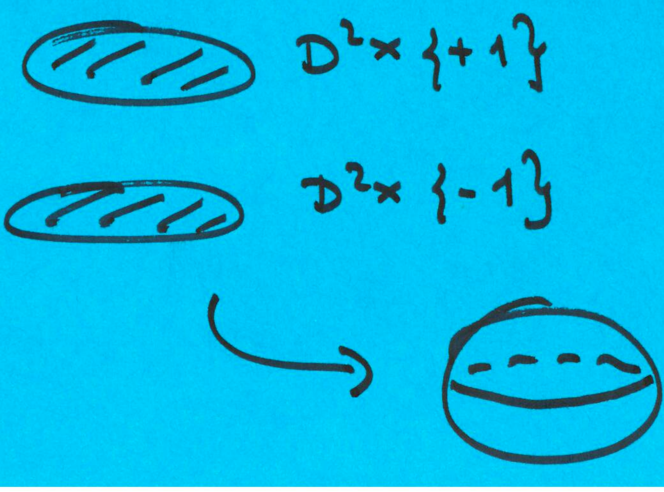
equivalence relation: $(0,t) \sim (1,t) \quad t \in [0,1]$

$(s,0) \sim (s,1) \quad s \in [0,1]$

square $\qquad$ torus

(c) In Hatcher you find 3 ways of obtaining S^2 (= a topological sphere) as a quotient.

$X = D^2 \times \{-1, 1\}$ "two copies of closed unit disk"



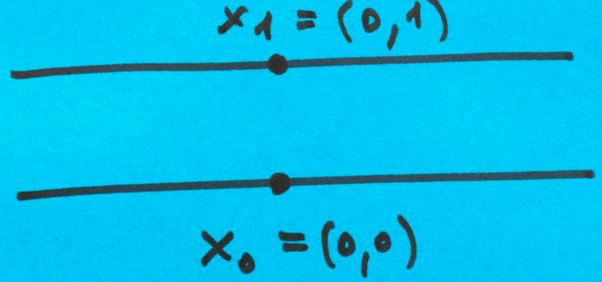
equivalence relation

$(x, 1) \sim (x, -1)$

for $x \in \partial D$

check: $X/\sim$ is homeo to S^2 unit sphere in $\mathbb{R}^3$

(d) <u>Achtung!</u> quotient of Hausdorff top. space may not be Hausdorff in general. (cf. problem 7.1). For instance:

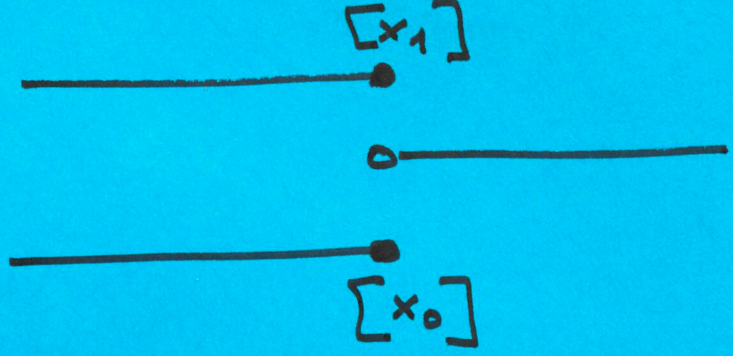


$X = \mathbb{R} \times \{0,1\}$

equivalence:

$(x,0) \sim (x,1) \quad x > 0$

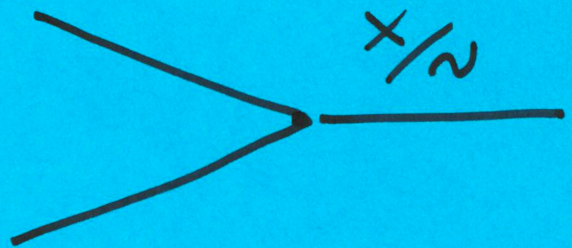
the result is a quotient space that looks like:



<u>check</u>: any non-empty open set containing $[x_0]$ will intersect any (non-empty) open set containing $[x_1]$

<u>rmk.</u> $X = \mathbb{R} \times \{0,1\}$ as above,

equivalence $(x,0) \sim (x,1) \quad x \geq 0$



$\cong$ cross of 3 lines in $\mathbb{R}^2$.

<u>Def.</u> Let X, Y be topological spaces. A <u>surjective</u> map $f: X \to Y$ is called a quotient map if

$$U \subset Y \text{ open} \iff f^{-1}(U) \subset X \text{ open}.$$

<u>review</u>: if X compact, Y Hausdorff then any continuous surjective map is a quotient map.

<u>Setup.</u>:
- $\longrightarrow$ a topological space X
- $\longrightarrow$ a set Y
- $\longrightarrow$ a surjective map $f: X \to Y$

Can we always place a topology on the target space Y so that $f: X \to Y$ is a quotient map?

<u>Answer</u>: YES, just <u>declare</u> that

$$U \subset Y \text{ is open} \iff f^{-1}(U) \subset X \text{ is open}.$$

In this case this topology is called (a) quotient topology on Y.

<u>Why this name</u>? claim: <u>there is nothing new</u>! $\longrightarrow$

Suppose we are in the situation above

$\longrightarrow$ a topological space X

$\longrightarrow$ a set Y

$\longrightarrow$ a surjective map $f: X \to Y$

- we can define an equivalence relation on X denoted $\sim$ by saying

$$x_1 \sim x_2 \iff f(x_1) = f(x_2)$$

In this case, at the level of sets there is an induced bijection

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "p"'] & & Y \\ & X/\sim \arrow[ur, dashed, "\bar{f}"'] & \end{tikzcd}$$

$$\bar{f}([x]) = f(x)$$

$\boxed{\bar{f}: X/\sim \longrightarrow Y \text{ bijection}}$

__Prop.__ $\bar{f}: X/\sim \longrightarrow Y$ is a __homeomorphism__

__Pf.__ • by the way we defined the quotient topology on $X/\sim$, have

$$V \subset X/\sim \text{ open} \iff p^{-1}(V) \subset X \text{ open}$$

- by the way we defined the quotient topology on Y, have

$$U \subset Y \text{ open} \iff f^{-1}(U) \subset X \text{ open}$$

Now: $f = \bar{f} \circ p \Rightarrow f^{-1}(U) = p^{-1}(\bar{f}^{-1}(U))$ (*)
$\forall\, U \subset Y$ open

i) $\bar{f}$ continuous $\Rightarrow f^{-1}(U)$ open
$\underset{\substack{p \text{ quotient} \\ \text{and use } (*)}}{\iff}$ $\bar{f}^{-1}(U)$ open
$\Rightarrow \bar{f}$ continuous

ii) take $V \subset X/_\sim$ open
and consider (*) for $U = \bar{f}(V)$.
Since $\bar{f}$ bijection, have

$$f^{-1}(\bar{f}(V)) = p^{-1}(V) \qquad (**)$$

p continuous $\Rightarrow p^{-1}(V)$ open
$\underset{\substack{f \text{ quotient} \\ \text{and use } (**)}}{\iff}$ $\bar{f}(V)$ open
$\Rightarrow \bar{f}$ open.

iii) i) + ii) $\Rightarrow \bar{f} : X/_\sim \rightarrow Y$ homeo. □

Continuity criterion for maps defined on a quotient:

<u>Prop.</u> Let $f : X \longrightarrow Y$ be a quotient map.
Then $g : Y \longrightarrow Z$ is continuous if and only if the composition $g \circ f : X \rightarrow Z$ is continuous.

Diagram :

$$\begin{tikzcd} X \arrow[rr, "g\circ f"] \arrow[dr, "f"'] & & Z \\ & Y \arrow[ur, "g"'] & \end{tikzcd}$$

Pf. one implication is trivial: if $g: Y \to Z$ is continuous then $g\circ f : X \longrightarrow Z$ is the composition of continuous maps and thus it is continuous (rmk.: $f: X \longrightarrow Y$ quotient $\Longrightarrow$ continuous)

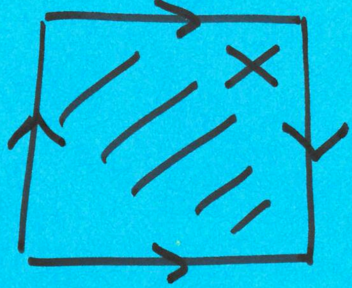
other implication: assume $g\circ f: X \longrightarrow Z$ continuous. Take $O \subset Z$ open

$$\underbrace{(g\circ f)^{-1}(O)}_{\text{open}} = f^{-1}(g^{-1}(O))$$

but f quotient means that $f^{-1}(V)$ is open $\Longleftrightarrow$ V is open, so apply this rmk to $V := g^{-1}(O)$ to conclude that $g^{-1}(O)$ is open for any $O \subset Z$ open. Hence g is continuous, like we had to prove. $\square$

Two geometric examples:

① the **Klein bottle**

start (as for the torus) w/ 

equivalence:

$(s, 0) \sim (s, 1) \quad s \in [0, 1]$

$(0, t) \sim (1, 1-t) \quad t \in [0, 1]$

Get an "abstractly defined" topological space $X/\sim$ called Klein bottle K^2.
It is a theorem that K^2 cannot be embedded in $\mathbb{R}^3$ i.e. there is no topological subspace in $\mathbb{R}^3$ homeomorphic to K^2.
(However, this would work in $\mathbb{R}^4$...)

② Projective Spaces

$\mathbb{P}^2(\mathbb{R})$, more generally one can consider $\mathbb{P}^d(\mathbb{K})$

Three ways of presenting it:

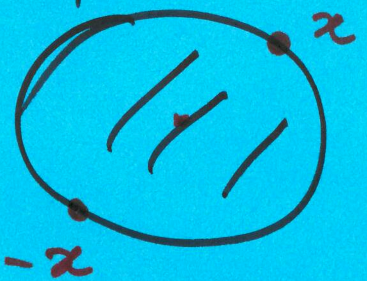
① $X = S^2$ 2-dim sphere (for us: unit sphere in $\mathbb{R}^3$ w/ subspace top.)

equivalence: $x \sim -x$ $\quad X/\sim \; =: \mathbb{P}^2(\mathbb{R})$

rmk. not embeddable in $\mathbb{R}^3$.

② $X = D^2$ 2-dim disk in $\mathbb{R}^2$

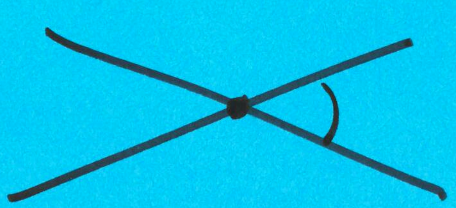
equivalence: for $x \in \partial D^2$ set $x \sim -x$



$X/\sim$ (homeo to) $\mathbb{P}^2(\mathbb{R})$

③ Consider the set of lines (through the origin) in $\mathbb{R}^3$, define a distance

$d(L_1, L_2) =$ "smallest angle between L_1, L_2" $\in [0, \pi/2]$



Claim: (lines, d) w/ induced topology $\underset{\text{homeo}}{\cong} \mathbb{P}^2(\mathbb{R})$

natural map: $q: S^2 \longrightarrow \mathbb{P}^2(\mathbb{R})$

$\qquad\qquad x \longmapsto L(x)$