Quotient Maps and Saturated Sets

recall: X, Y top. spaces.

$f: X \to Y$ is a quotient map if it is surjective and

(*) $U \subset Y$ open $\iff f^{-1}(U) \subset X$ open.

Comments:

① a quotient map is continuous

why? "continuity" means $\boxed{\Rightarrow}$ in (*)

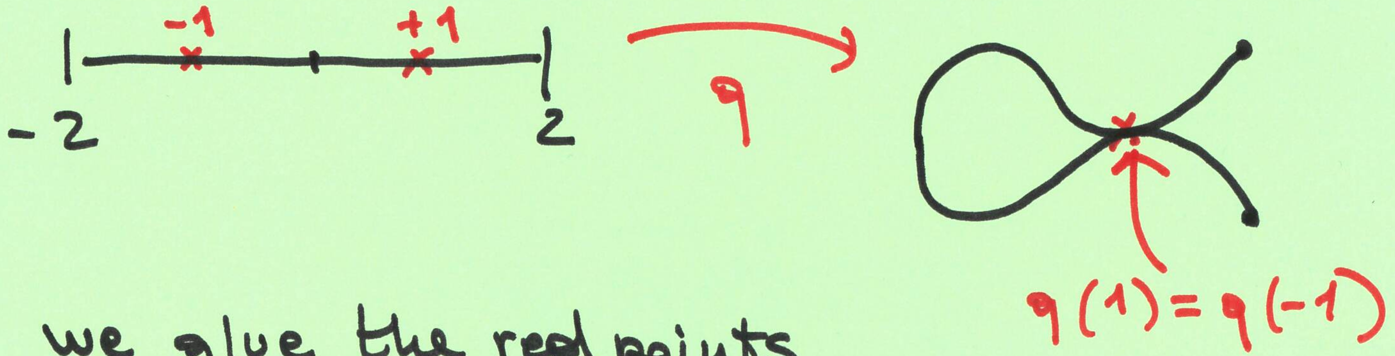
② a quotient map is not open (in general)

i.e. the implication $\boxed{\Leftarrow}$ in (*) does not imply openness in general.

An example:

$X = [-2, 2]$ w/ Euclidean top.

equivalence relation $\{-1\} \sim \{+1\}$

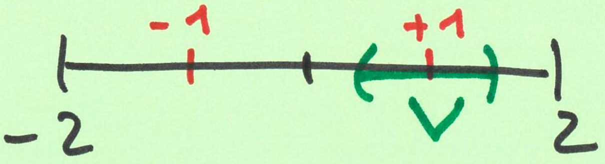


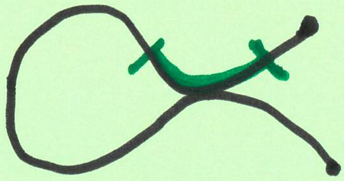
we glue the red points together.

<u>Claim</u>: $q : X \longrightarrow X/\sim$ is not open.

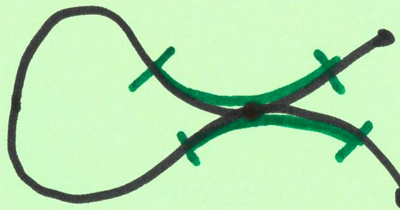
take $V = (1/2, 3/2) \subset X$

$q(V)$ is <u>not</u> open in $X/\sim$. why?





morally speaking: each open set in $X/\sim$ containing $q(1) \equiv q(-1)$ should rather look like



formally speaking: if $U := q(V)$ were open then (since q is continuous) $q^{-1}(U) = q^{-1}(q(V))$ would be open in $X = [-2, 2]$

but on the other hand

$$q^{-1}(q(V)) = \underbrace{\{-1\} \cup (1/2, 3/2)}_{\text{not open in } X} \qquad \square$$

q.: when is a quotient map $f: X \to Y$ an open map?

recall (again):

<u>quotient</u> (means surjective and)

(*) $U \subset Y$ open $\iff f^{-1}(U) \subset X$ open

<u>open</u>

(**) $V \subset X$ open $\implies f(V) \subset Y$ open

<u>Def.</u> X, Y sets, $f: X \to Y$ map.

(i) given $V \subset X$ subset we call <u>saturation</u>, of V, the set $\tilde{V} := f^{-1}(f(V))$

(<u>rmk.</u> always true $f^{-1}(f(V)) \supset V$ but this is a strict inclusion in general)

(ii) we say that $V \subset X$ subset is <u>saturated</u> if $\tilde{V} = V$ (i.e. if V coincides with its saturation.

<u>Example</u>: $X = \{1, 2, 3, 4\}$ $Y = \{a, b, c\}$

$f: X \to Y$ defined by the diagram

$V = \{2\}$

$\tilde{V} = \{1, 2\}$

The implication $\boxed{\Leftarrow}$ in $(*)$ only implies that $(**)$ holds for <u>saturated sets</u>, i.e. what we can always conclude is

$$(**)_{sat} \quad \left.\begin{array}{l} V \subset X \text{ open} \\ V \text{ saturated} \end{array}\right\} \Rightarrow f(V) \subset Y \text{ open}$$

which is strictly weaker than openness in general.

<u>Rmk.</u>: $(**)_{sat}$ coincides w/ $(**)$

$$\Updownarrow$$

$(\#)$ $\forall\, V \subset X$ open is saturated, i.e. $V = \tilde{V}$.

Condition $(\#)$ depends both on

- $\longrightarrow$ the topology on X
- $\longrightarrow$ the set-theoretic properties of f.

<u>Rmk.</u>: if $f: X \longrightarrow Y$ is a bijection then $f^{-1}(f(V)) = V$ for all $V \subset X$ subset so a bijective quotient map is <u>open</u> and thus a homeomorphism.

<u>next</u>: what to do w/ non-injective maps?

<u>Openness Criterion</u>: let X, Y be top. spaces and let $f: X \to Y$ be a quotient map. Then:

f is open $\iff$ $\forall\, V \subset X$ open have $f^{-1}(f(V))$ <u>open</u>.

(So: if I can find $V \subset X$ open, whose saturation is not open then I can conclude that $f: X \to Y$ is not open.)

<u>Pf.</u>: $\boxed{\Rightarrow}$ is <u>easy</u> because if f is open then $f(V) \subset Y$ open and (continuity) $f^{-1}(f(V))$ is open in X.

$\boxed{\Leftarrow}$ here we just use condition (*) for quotient maps

$U \subset Y$ open $\iff$ $f^{-1}(U) \subset X$ open

Apply this to $U = f(V)$. Know, by assumption that $f^{-1}(f(V)) = f^{-1}(U)$ is open, but then $f(V)$ is open in Y. This argument applies to any given $V \subset X$ open, which means that the map f is open.

$\square$