Topological Manifolds - Lecture 14

Last week we defined (as a quotient)

the torus T^2 

it looks different than (say)

the sphere S^2 

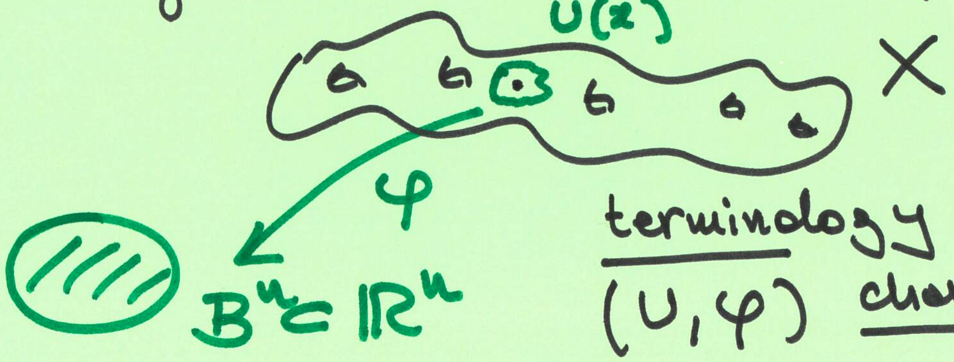
but both are special incarnations of the same Mathematical object.

<u>Def.</u> A top. space X is called a topological n-dimensional manifold if:

a) it is Hausdorff

b) any $x \in X$ has an open neighbourhood $U(x)$ that is homeomorphic to $\mathbb{R}^n$.

<u>rmk.</u> recall that for any $n \geq 1$, one has that $\mathbb{R}^n$ is homeomorphic to the open ball $B^n = \{x \in \mathbb{R}^n : |x| < 1\}$, so b) could be replaced by "any $x \in X$ has an open neigh. $U(x)$ that is homeomorphic to B^n.



<u>terminology:</u>
(U, φ) <u>chart</u> for x

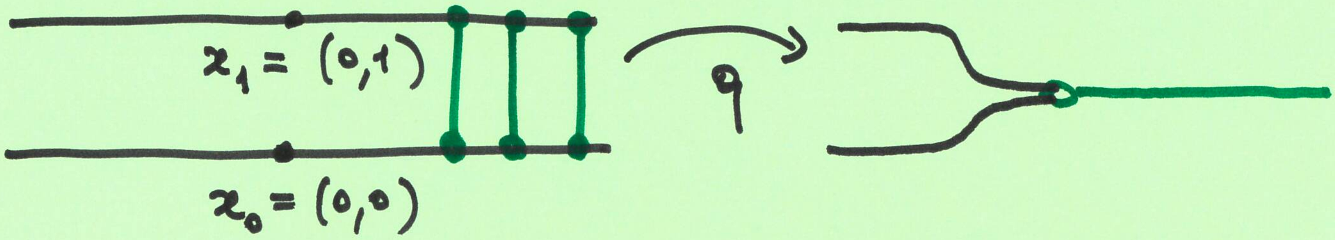
<u>rmk. 2</u> it follows from the definition that a top. n-manifold is locally path-connected hence (cf. pg. 25) the path-connected comp. coincide with the connected comp.
(the two partitions coincide!)
Also, it follows that each connected component is <u>open</u>. Thus, if an n-manifold is compact then it has only <u>finitely many</u> connected comp.
(why? connected comp. provide an open cover).

<u>rmk. 3</u> in the previous def.

$$ⓑ \not\Longrightarrow ⓐ$$

For instance: $X = \mathbb{R} \times \{0,1\}$, equivalence $(x,0) \sim (x,1)$ for $x > 0$

Fact: the quotient $X/\sim$ has property ⓑ but it is not Hausdorff.



$[x_0] \neq [x_1]$ but one cannot find disjoint open neighborhoods.

Examples:

① $S^n := \{x \in \mathbb{R}^{n+1} : |x| = 1\}$ $n \geq 1$

a topological n-manifold

a) Hausdorff (subspace of $\mathbb{R}^{n+1}$, which is H.)

b) first: the north pole $\tilde{x} = (0, \dots, 0, 1)$ has a neighborhood homeo to unit ball in $\mathbb{R}^n$.

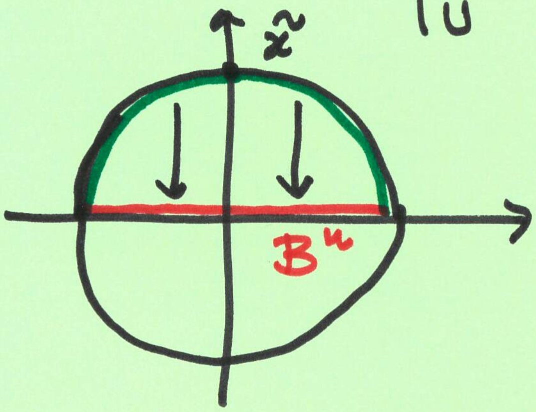
$\mathbb{R}^{n+1} = \mathbb{R}^n \times \mathbb{R}$, $\pi : S^n \longrightarrow \mathbb{R}^n$

$(x_1, \dots, x_{n+1}) \longmapsto (x_1, \dots, x_n)$

take $U = \{x \in S^n : x_{n+1} > 0\}$

open upper hemisphere

check: $\pi|_U : U \longrightarrow B^n$ is a homeo.

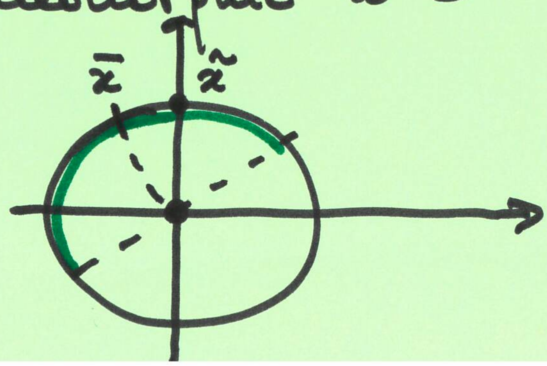


picture for $\boxed{n=1}$

how about the other points? given $\bar{x} \in S^n$ let $\sigma_{\bar{x}} \in SO(n+1)$ i.e. an isometry of $\mathbb{R}^{n+1}$ such that $\sigma_{\bar{x}}(\tilde{x}) = \bar{x}$.

Then $\sigma_{\bar{x}}(U)$ is an open neighborhood of $\bar{x} = \sigma_{\bar{x}}(\tilde{x})$, that is homeomorphic to B^n by means of the map

$\pi \circ \sigma_{\bar{x}}^{-1}$



② $\mathbb{P}^n(\mathbb{R})$ ← n-dimensional top. manifold

(we focus on $n=2$ for simplicity)

$\mathbb{P}^2(\mathbb{R}) = S^2/\sim$ equivalence $x \sim -x$ (identification of antipodal points)

Given $[x] \in \mathbb{P}^2(\mathbb{R})$, if $p: S^2 \longrightarrow \mathbb{P}^2(\mathbb{R})$ $p^{-1}([x]) = \{x, -x\}$. Consider the open half-sphere $\underbrace{\sigma_x(U)}_{V_x} \subset S^2$. Then:

- $p(V_x)$ is an open set in $\mathbb{P}^2(\mathbb{R})$, cont. $[x]$

(why? p is a quotient map, hence

$p(V_x)$ is open in $\mathbb{P}^2(\mathbb{R}) \iff p^{-1}(p(V_x))$ is open in S^2

but $p^{-1}(p(V_x))$ is the whole sphere S^2 minus an equatorial circle $C(x)$, in fact

$\underbrace{p^{-1}(p(V_x))}_{\text{open in } S^2} = \{z \in S^2 : \langle z, x \rangle \neq 0\}$ property ⓑ is OK.

$p(V_x)$ is homeomorphic to B^2

(why? note that $p|_{V_x} : V_x \longrightarrow \mathbb{P}^2(\mathbb{R})$ is a homeomorphism onto its image, but we already know that $V_x \underset{\text{homeo}}{\cong} B^2$.)

• we must still check that $\mathbb{P}^2(\mathbb{R})$ is Hausdorff.

Given $[x_1] \neq [x_2]$ let

$\lambda = \min \{ |x_1 - x_2|, |x_1 + x_2|, \pi \}$

set $\delta = \lambda/10$ and $\begin{cases} U_1 = B^{\mathbb{R}^3}(x_1, \delta) \cap S^2 \\ U_2 = B^{\mathbb{R}^3}(x_2, \delta) \cap S^2 \end{cases}$

then $U_i \subset V_{x_i}$ $i = 1,2$ so $p|_{U_i} : U_i \to \mathbb{P}^2(\mathbb{R})$ is the restriction of a homeomorphism, in part. $p(U_i)$ is open in $\mathbb{P}^2(\mathbb{R})$, contain $[x_i]$, $i = 1, 2$. Also, by construction

$p(U_1) \cap p(U_2) = \emptyset \Rightarrow \mathbb{P}^2(\mathbb{R})$ is T_2

(property (a) is OK).

<u>Other examples:</u> • the torus $S^1 \times S^1$

• the Klein bottle K^2

<u>Terminology</u>: a 2-dimensional top. manifold is also called (top.) surface.

Hence: S^2, $\mathbb{P}^2(\mathbb{R})$, T^2, K^2 are surfaces.

They are examples of compact surfaces (i.e. surfaces that are compact top. spaces)

<u>Achtung!</u> $\mathbb{R}^2$ is an example of a (non-comp.) (topological) surface; more generally take the graph of any continuous fct. $f: \mathbb{R}^2 \to \mathbb{R}$.

<u>Big question</u>: can we classify all (compact) n-dimensional top. manifolds up to homeomorphism?

<u>Morally speaking</u>: complexity grows "exponentially" w.r.t. the dimension n.

<u>Case $\boxed{n=1}$</u>:

<u>Thm.</u> A compact, connected 1-dimensional top. manifold is homeomorphic to S^1.

<u>Cor.</u> $\mathbb{P}^1(\mathbb{R})$ is homeo to S^1 (not obvious from the def. of $\mathbb{P}^1(\mathbb{R})$).

<u>Pf.</u> 1) each point $x \in X$ has an open nbh. $U(x)$ that is homeomorphic to $B^1 = (-1,1)$

standard open interval in $\mathbb{R}$

2) by compactness take an open cover $U_1, U_2, \ldots, U_n$ and wlog can assume this cover is <u>not</u> redundant, meaning that $U_i \not\subseteq U_j$ if $i \neq j$.

3) proceed by induction on n. It must be $\boxed{n \geq 2}$ since for $n=1$ $X = U_1$

compact   not compact

4) since X is connected there must be indices $i \neq j$ w/ $U_i \cap U_j \neq \emptyset$

5) <u>key claim</u>: In the setting above, if $U_1 \cap U_2 \neq \emptyset$

either (a) $U_1 \cup U_2 \underset{\text{homeo}}{\cong} S^1$

or (b) $U_1 \cup U_2 \underset{\text{homeo}}{\cong} B^1$

Given this claim, let's see how to conclude.

6) $\boxed{n = 2}$ <u>For case (a)</u>: note that $U_1 \cup U_2 = X$

$(\Rightarrow X \underset{\text{homeo}}{\cong} S^1)$

<u>For case (b)</u>: this cannot happen because X is compact, B^1 is not.

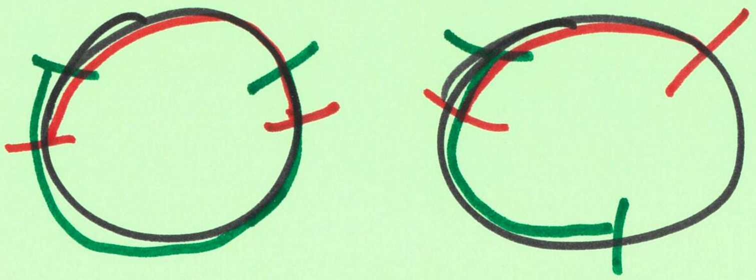
$\boxed{n > 2}$ <u>For case (a)</u>: note that $U_1 \cup U_2$ is open in X, but also closed in X

($U_1 \cup U_2$ <u>compact</u> ($\cong S^1$) in X, which is T_2, hence closed)

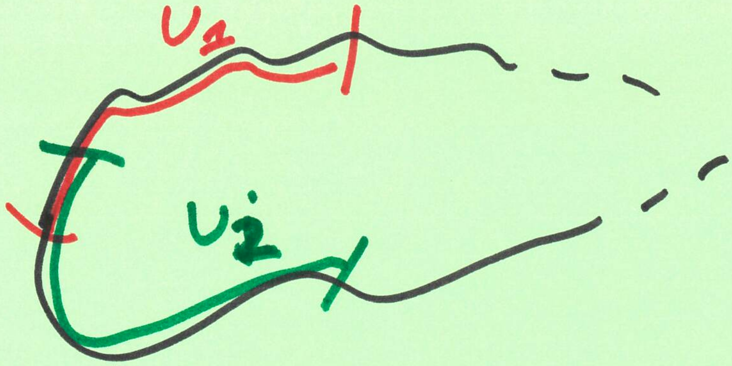
$\Longrightarrow X = U_1 \cup U_2 \cong S^1$

<u>For case (b)</u>: just replace $\{U_1, U_2\}$ by $U_1 \cup U_2$ in the cover, and invoke induction.

7) let's prove the claim:



<u>goal</u>: the picture on the left is universal (nothing else can happen!)



$U_1 \cap U_2$ open in X hence (by def. of subspace top.) also open in U_1 (or U_2)

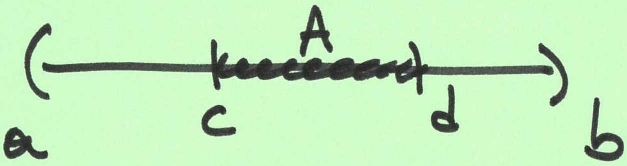
so $U_1 \cap U_2$ is a union of open intervals, let A be a connected comp. of such intersection.

Then: A must be an "end interval of U_1"

i.e. $U_i \longrightarrow (a, b)$

$\varphi(A) = (a, c)$ <u>or</u> $\varphi(A) = (c, b)$

for some $a < c < b$. If not $a < c < d < b$

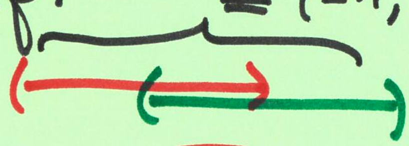


$\Longrightarrow U_1 \subset U_2$ or $U_2 \subset U_1$

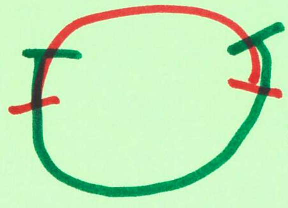
violates non redundancy of the cover

But (since an interval has only 2 endpoints) we then have that $U_1 \cap U_2$ consists of either one or two conn. comp. $\cong (-1, 1)$

If $|U_1 \cap U_2| = 1$ then



If $|U_1 \cap U_2| = 2$ then

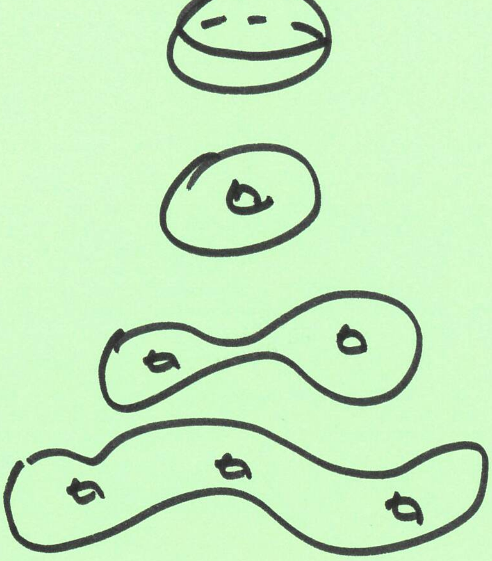


$\cong S^1$

$\square$

The problem of classifying compact connected surfaces ($n = 2$) is not trivial.

<u>Picture:</u>



orientable	non-orientable	
	$\mathbb{P}^2(\mathbb{R})$	$g=0$
	K^2	$g=1$
	.....	$g=2$
	.....	$g=3$
.....	....	....

<u>Theorem</u>: any compact (connected) surface is homeomorphic to an element of the list above.

<u>Two parts</u>:
- any two models in the list are pairwise non homeomorphic (list above is minimal)
- any given compact surface X is homeo to an element of the list.

[AT] serves this scope!