Homotopy - Lecture 15

Algebraic Topology:
we'll describe the construction of the fundamental group of a top. space as a case study.

Idea:

$$\{\text{top. spaces}\} \longrightarrow \{\text{groups}\}$$

$\boxed{\text{machine}}$

$$\{\text{continuous maps}\} \longrightarrow \left\{\begin{matrix}\text{group}\\ \text{homomorphism}\end{matrix}\right\}$$

so we want to associate

$$X \rightsquigarrow \pi_1(X)$$

$$f: X \to Y \rightsquigarrow f_*: \pi_1(X) \to \pi_1(Y)$$

Formally, we want to build a functor from the category of (top. spaces, cont. maps) to the category of (groups, group. homom.).

Basic requirements:

i) $(\mathrm{Id}_X)_* = \mathrm{Id}_{\pi_1(X)}$

ii) $(g \circ f)_* = g_* \circ f_*$

Specific requirement:

$$X_1 \underset{\text{homeo}}{\cong} X_2 \Rightarrow \pi_1(X_1) = \pi_1(X_2)$$

There is a weaker, but more useful, equivalence relation one can place on topological spaces, which is called <u>homotopy</u>.

<u>Def.</u> Let X be a top. space, let $A \subset X$ be a top. subspace (possibly $A = \emptyset$); let Y be a top. space.
We say that two continuous functions $f_0, f_1 : X \longrightarrow Y$ are homotopic (relative to A) if $\exists$ $F : X \times I \longrightarrow Y$ continuous, such that

i) $F(x, 0) = f_0(x)$
$F(x, 1) = f_1(x)$ $(\forall x \in X)$

ii) $F(a, t) = f_0(a) = f_1(a)$
$\forall a \in A$, $\forall t \in [0, 1]$.

If $A = \emptyset$ then one drops ii) and we just say that f_0, f_1 are homotopic.

<u>Example</u>: homotopy of paths
$I = [0, 1]$ $\leftsquigarrow$ plays the role of X
$A = \{0, 1\}$
target space: a top. space Z
A path is a continuous map $\gamma : I \longrightarrow Z$

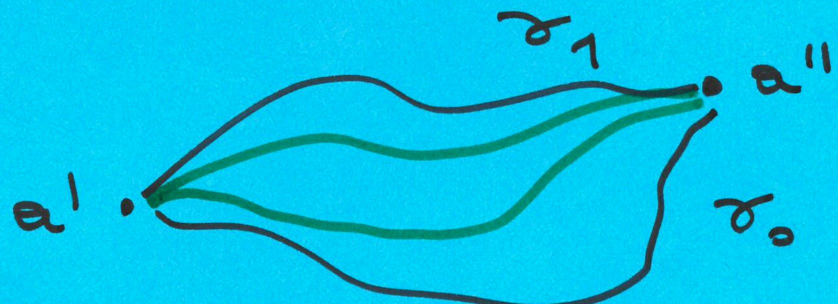
Given two paths $\gamma_0, \gamma_1 : I \longrightarrow Z$
w/ same endpoints i.e. $\gamma_0(0) = \gamma_1(0)$
$\gamma_0(1) = \gamma_1(1)$

a homotopy is a continuous family of paths from γ_0 to γ_1, i.e. $H : I \times I \longrightarrow Z$

w/ (i) $H(s,0) = \gamma_0(s)$
$H(s,1) = \gamma_1(s)$

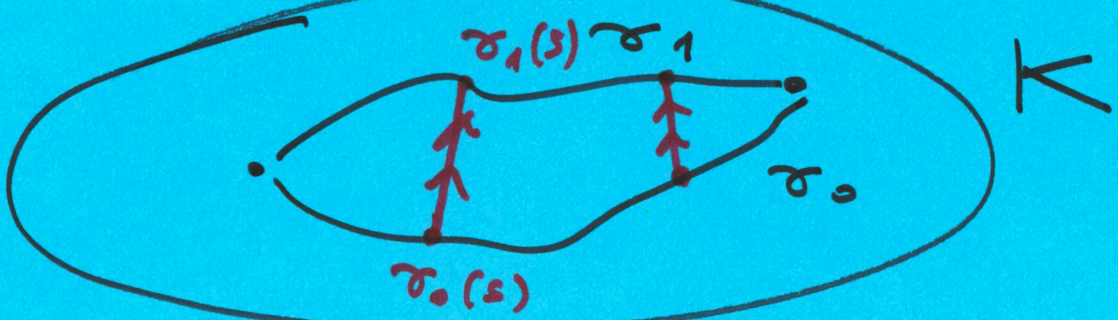
(ii) $H(0,t) = \gamma_0(0) = \gamma_1(0)$
$H(1,t) = \gamma_0(1) = \gamma_1(1)$

Picture :



Rmk. let $K \subseteq \mathbb{R}^n$ be convex (possibly $K = \mathbb{R}^n$). Then: any two paths γ_0, γ_1 w/ the same endpoints are homotopic via the linear homotopy

$$H(s,t) = (1-t)\gamma_0(s) + t\gamma_1(s).$$



<u>Prop.</u> In the setting of the previous def. the homotopy of maps (possibly: relative to A) is an equivalence relation.

<u>Pf.</u> a) $f \simeq f$: take $F(x,t) = f(x)$
(keep still)

b) $f \simeq g$, $\exists\, F : X \times I \to Y$
w/ $F(x,0) = f$
$F(x,1) = g$ (plus ii)

$\downarrow$ (?)

$g \simeq f$ set $\tilde{F} : X \times I \to Y$
$\tilde{F}(x,t) = F(x, 1-t)$
(invert time arrow)

c) $\underbrace{f \simeq g}_{F}$, $\underbrace{g \simeq h}_{G}$ $\xrightarrow{(?)}$ $f \simeq h$

A homotopy from f to h is just

$$H(x,t) = \begin{cases} F(x, 2t) & 0 \leq t \leq \frac{1}{2} \\ G(x, 2t-1) & \frac{1}{2} \leq t \leq 1 \end{cases}$$

Claim: since $F(x,1) = G(x,0) \equiv g(x)$ then $H : X \times I \to Y$ is a continuous map.

<u>why ?.</u> let X, Y be top. spaces and assume $X = A \cup B$ w/ $A, B \subset X$ <u>closed</u>.

If $f_1 : A \to Y$, $f_2 : B \to Y$ coincide on $A \cap B$, then $f : X \to Y$ given by $f(x) = \begin{cases} f_1(x) & A \\ f_2(x) & B \end{cases}$ is continuous.

(take $C \subset Y$ closed, $f^{-1}(C) = \underbrace{f_1^{-1}(C)}_{\text{closed in } A} \cup \underbrace{f_2^{-1}(C)}_{\text{closed in } B}$

but A, B closed in X $\quad\Downarrow$ $f_1^{-1}(C)$ closed in X $\quad\Downarrow$ - - .

$\Rightarrow f^{-1}(C) \subset X$ closed $\Rightarrow f$ continuous). $\square$

Notation: $[f]$ equivalence class of $f : X \to Y$ (rel A)

special case $X = I$, $A = \{0,1\}$ $\quad [\gamma] \leftarrow$ homotopy class of γ relative to endpoints

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Homotopy of top. spaces

Def. Two top. spaces X, Y are called homotopic if $\exists$ $f : X \to Y$ and $g : Y \to X$ such that
$$g \circ f \simeq \mathrm{id}_X$$
$$f \circ g \simeq \mathrm{id}_Y .$$

Rmk. If $X \underset{\text{homeo}}{\cong} Y$ then $X \simeq Y$.

<u>Prop.</u> This is an equivalence rel. in the class of top. spaces.

<u>Lemma:</u>

$$\begin{tikzcd} Z_1 \arrow[r, shift left, "f"] \arrow[r, shift right, "f'"'] & Z_2 \arrow[r, shift left, "g"] \arrow[r, shift right, "g'"'] & Z_3 \end{tikzcd}$$

If $\underbrace{f \simeq f'}_{F}$ and $\underbrace{g \simeq g'}_{G}$ then $g\circ f \simeq g'\circ f'$.

<u>Pf.</u> Set $H : Z_1 \times I \longrightarrow Z_3$ by

$H(z_1, t) = G\big(F(z_1, t), t\big)$. Now

- $t = 0 \quad H(z, 0) = G(f(z), 0) = g \circ f$
- $t = 1 \quad H(z, 1) = \dots = g' \circ f'$

<u>Continuous:</u> H is the composition of

- $(z_1, t) \longmapsto (F(z_1, t), t)$

  $\in Z_1 \times I \qquad\qquad Z_2 \times I$

  continuous by componentwise criterion

- $G : Z_2 \times I \longrightarrow Z_3$ $\square$

Now, let's get back to the prop.:

$$\begin{tikzcd} X \arrow[r, bend left, "f"] & Y \arrow[l, bend left, "g"] \arrow[r, bend left, "f'"] & Z \arrow[l, bend left, "g'"] \end{tikzcd}$$

Know: $\begin{cases} g\circ f \simeq id_X \\ f\circ g \simeq id_Y \end{cases}$ $\begin{cases} g'\circ f' \simeq id_Y \\ f'\circ g' \simeq id_Z \end{cases}$

Want: $g\circ g'\circ f'\circ f \simeq id_X$

$f'\circ f\circ g\circ g' \simeq id_Z$

1st check:

$$g\circ \underbrace{g'\circ f'}_{\simeq id_Y}\circ f \underset{(\text{Lemma})}{\simeq} g\circ id_Y\circ f \equiv \underbrace{g\circ f}_{\simeq id_X} \qquad \square$$

Example: a top. space X is contractible if it is homotopic to a point.

Fact: a non-empty convex $K\subset \mathbb{R}^n$ is contractible (in part. $\mathbb{R}^n$, D^n, B^n $\forall n\geq 1$)

Why? $K\neq\emptyset \Rightarrow \exists x_0\in K$ consider

$$\begin{tikzcd} \{x_0\} \arrow[r, bend left, "\iota"] & K \arrow[l, bend left, "\rho"] \end{tikzcd} \qquad \iota(x_0)=x_0, \quad \rho(x)=x_0$$

$\rho\circ\iota = id_{\{x_0\}}$ $\quad \iota\circ\rho \simeq id_X$ $\quad H(x,t)=(1-t)x+tx_0$