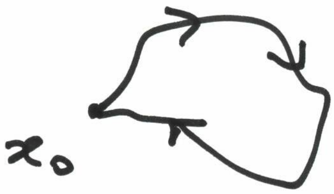
The fundamental group - Lecture 16

X top. space $(\neq \emptyset)$

Ψ

x_0 basepoint

- Loops based at x_0: any $\gamma : I \to X$ s.t. $\gamma(0) = \gamma(1) = x_0$



$\Omega(X, x_0)$

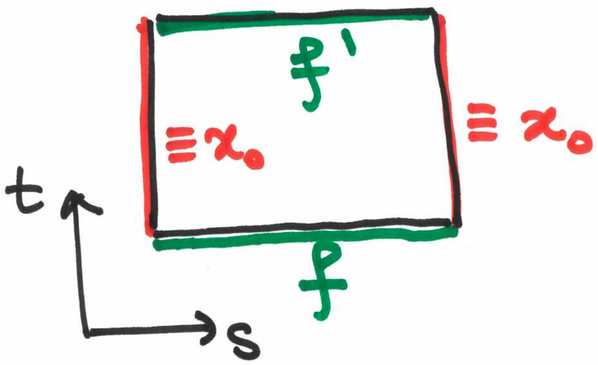
- Homotopy of loops:

$f, f' \in \Omega(X, x_0)$ we say $f \simeq f'$

if $\exists\ F : I \times I \to X$ w/

$F(s, 0) = f(s)$, $F(s, 1) = f'(s)$ $\forall s$

$F(0, t) = x_0$, $F(1, t) = x_0$ $\forall t$



- Concatenation of loops:

given $\left.\begin{array}{l} f : I \to X \\ g : I \to X \end{array}\right\} \Longrightarrow f \cdot g : I \to X$

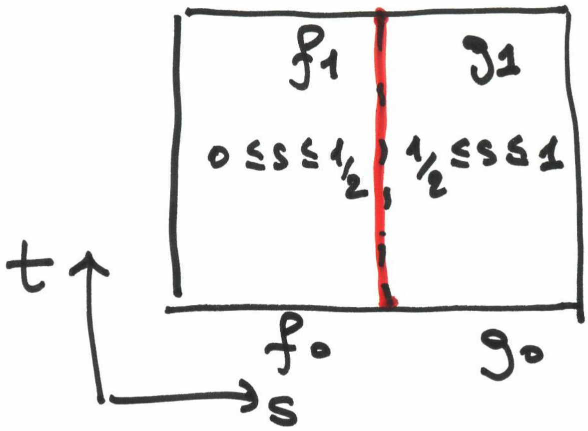
<u>loops</u>

$$f \cdot g(t) = \begin{cases} f(2t) & 0 \leq t \leq 1/2 \\ g(2t-1) & 1/2 \leq t \leq 1 \end{cases}$$

• Concatenation and Homotopy of loops:

$$\left.\begin{array}{l} \text{If } \; f_0 \simeq f_1 \\ \quad\; g_0 \simeq g_1 \end{array}\right\} \Rightarrow f_0 \cdot g_0 \simeq f_1 \cdot g_1$$

why? $H(s,t) = \begin{cases} F(2s,t) & 0 \leq s \leq 1/2 \\ G(2s-1,t) & 1/2 \leq s \leq 1 \end{cases}$



for $s = 1/2$ the definitions agree $F(1,t) = G(0,t) = x_0$ $\forall\, t \in [0,1]$, so invoke continuity criterion seen L15.

<u>Moral</u>: the operation $[f]\cdot[g] := [f\cdot g]$ is well-defined.

<u>Def.</u> we define the fundamental group of a top. space X w.r.t. $x_0 \in X$ as the set of all homotopy classes of loops based at x_0 with operation $[f]\cdot[g] := [f\cdot g]$.

<u>Notation</u>: $\pi_1(X, x_0)$

Comments: 1) later we'll discuss role of basepoint

2) $\pi_1(X,x_0)$ is the 1^{st} in the sequence of homotopy groups $\pi_n(X,x_0)$ $n \geq 1$ ($n=0$ is a bit special...)

Example: let $K \subseteq \mathbb{R}^4$ be convex, $x_0 \in K$ (e.g. $\mathbb{R}^4, B^4, D^4 \dots$). $\pi_1(K, x_0) = \{0\}$

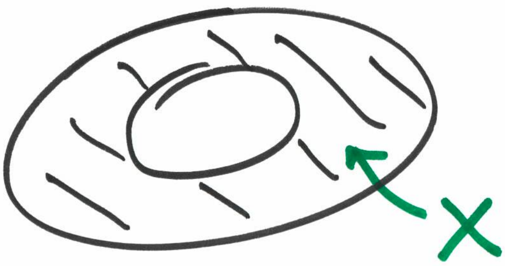
why? any two loops $f, g : I \to K$ are homotopic via a linear homotopy

$$H(s,t) = (1-t) f(s) + t g(s)$$

So any given loop is homotopic to the trivial loop $c_{x_0} : I \to K$ $c_{x_0}(s) = x_0$.

$\Rightarrow$ there is only one equivalence class i.e. $\pi_1(K, x_0)$ is the group w/ one element.

Achtung! we'll see that $\boxed{\pi_1(S^1, z_0) \cong \mathbb{Z}}$ so if we took a (non-convex) domain w/ a hole



$\pi_1(X, x_0) \cong \mathbb{Z}$

Def. A top. space X is called simply-connected if

a) it is path-connected

b) it has trivial fundamental group

(example above: $\mathbb{R}^4, B^4, D^n \dots$)

<u>Pf.</u> ($\pi_1(X, x_0)$ is well-defined i.e. the axioms of group are satisfied).

<u>trick</u>: given continuous $\varphi: I \circlearrowleft$ $\begin{cases} \varphi(0) = 0 \\ \varphi(1) = 1 \end{cases}$

the reparametrization of $f: I \longrightarrow X$ via φ is just $f \circ \varphi(s) = f(\varphi(s))$.

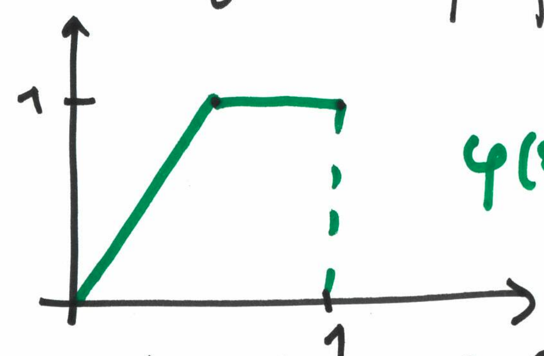
fact: $f \circ \varphi \simeq f$ via the homotopy $F: I \times I \longrightarrow X$ given by

$$F(s,t) = f((1-t)\varphi(s) + ts) \quad \square$$

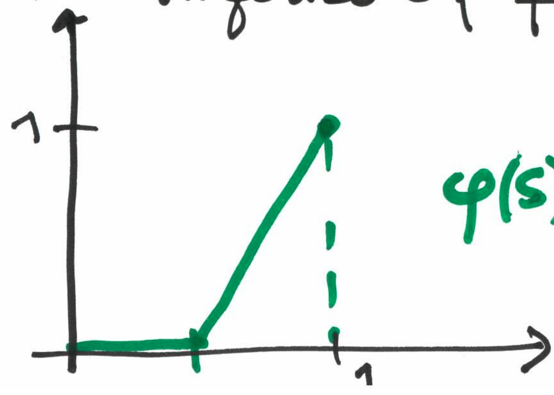
(a) identity element $[c_{x_0}]$.

must check $\begin{cases} f \cdot c \simeq f \\ c \cdot f \simeq f \end{cases}$ in $\Omega(X, x_0)$

$f \cdot c$ is a reparametrization of f

via  $\varphi(s) = \begin{cases} 2s & 0 \le s \le \frac{1}{2} \\ 1 & \frac{1}{2} \le s \le 1 \end{cases}$

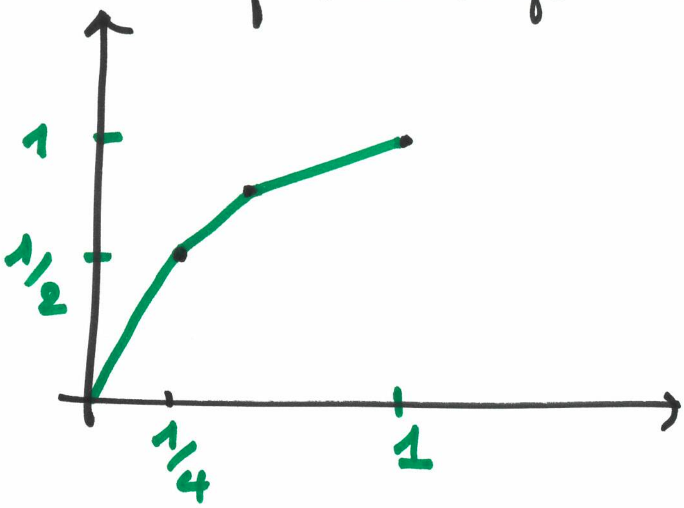
$c \cdot f$ is a reparametrization of f

via  $\varphi(s) = \begin{cases} 0 & 0 \le s \le \frac{1}{2} \\ 2s-1 & \frac{1}{2} \le s \le 1 \end{cases}$

(b) associative: $(f\cdot g)\cdot h \simeq f\cdot(g\cdot h)$

Note that $f\cdot(g\cdot h)$ is a reparametrization of $(f\cdot g)\cdot h$ via

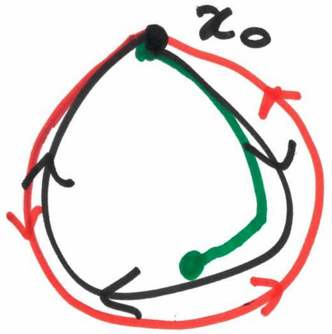
$$\varphi(s)=\begin{cases} 2s & 0\le s\le 1/4 \\ s+1/4 & 1/4\le s\le 1/2 \\ \frac{s}{2}+1/2 & 1/2\le s\le 1 \end{cases}$$

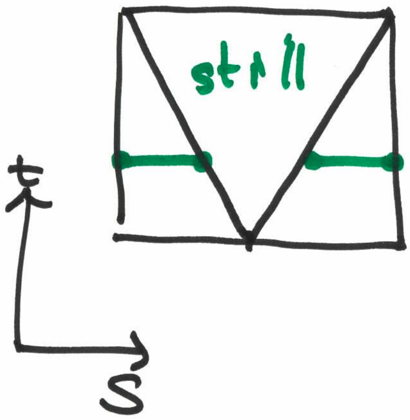


(c) <u>inverse element</u>: given $f: I\to X$

define $\bar f(s)=f(1-s)$ [time inversion]

Claim: $f\cdot\bar f\simeq c$ (similarly $\bar f\cdot f\simeq c$)





$H: I\times I\to X$

$$H(s,t)=\begin{cases} f(2s) & s\le \frac{1-t}{2} \\ f(1-t) & \frac{1-t}{2}\le s\le\frac{1+t}{2} \\ \bar f(2s-1) & s\ge\frac{1+t}{2} \end{cases}$$

H is continuous thanks to continuity criterion we saw in L15 (here for 3 closed sets)

$t=0 \qquad H(s,0)=f\cdot\bar f$

$t=1 \qquad H(s,1)\equiv x_0$ $\square$

<u>Rmk</u>: except in very special cases $\pi_1(X,x_0)$ is <u>not</u> abelian.

<u>Technical remark</u>: we can define the <u>n-fold</u> concatenation of paths. Setup: X top. space, $x_0 \in X$ basepoint, $n \in \mathbb{N}_*$. $f_1, \dots, f_n : I \to X$

$$\begin{cases} f_1(0) = x_0 \\ f_n(1) = x_0 \end{cases} \quad \text{and} \quad f_i(1) = f_{i+1}(0) \quad i = 1, \dots, n-1$$

then set
$$f_1 \cdot \dots \cdot f_n(s) = \begin{cases} f_1(ns) & 0 \le s \le 1/n \\ f_2(ns-1) & 1/n \le s \le 2/n \\ \dots \end{cases}$$

<u>Fact 0</u>: suppose $\forall i \in \{1, \dots, n\}$ $\exists$ homotopy $F_i : I \times I \longrightarrow X$ w/

a) $F_i(s, 0) = f_i$, $F_i(s, 1) = f_i'$

b) $F_1(0, t) = x_0$, $F_{n-1}(1, t) = x_0$ and $F_i(1, t) = F_{i+1}(0, t)$, $i \in \{1, \dots, n-1\}$

then $f_1 \cdot f_2 \cdot \dots \cdot f_n \simeq f_1' \cdot f_2' \cdot \dots \cdot f_n'$

homotopy on loops in $\Omega(X, x_0)$

<u>Fact 1</u>: in the setting above, using piecewise linear reparametrizations one has

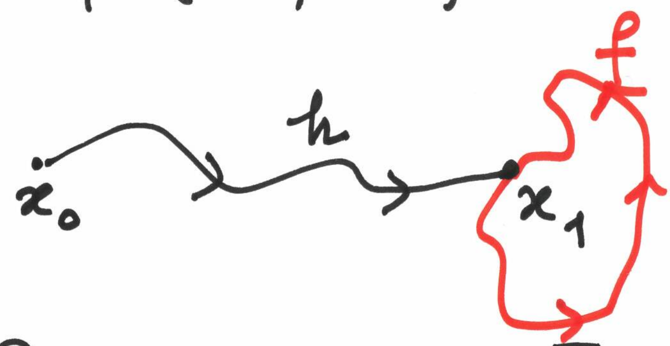
$\boxed{n=3}$ $(f_1 \cdot f_2) \cdot f_3 \simeq f_1 \cdot (f_2 \cdot f_3) \simeq f_1 \cdot f_2 \cdot f_3$

more generally one can place arbitrary parentheses

$f_1 \cdot \dots \cdot f_n \simeq (f_1 \cdot f_2) \cdot f_3 \cdot \dots \cdot f_n \simeq (f_1 \cdot f_2 \cdot f_3) \cdot \dots \cdot f_n$

<u>Role of basepoint</u>: note that $\pi_1(X, x_0)$ only contains info about the path-connected comp. of X containing x_0.

If x_0, x_1 belong to the same path-connected comp. what is the relation between $\pi_1(X, x_0)$ and $\pi_1(X, x_1)$?



$h : I \longrightarrow X$ path connecting $h(0) = x_0$ w/ $h(1) = x_1$.
$\bar{h} : I \longrightarrow X$ be the inverse path.

$f \rightsquigarrow h \cdot f \cdot \bar{h}$

f based at x_1; $h \cdot f \cdot \bar{h}$ based at x_0

rmk. :	$h \cdot \bar{h} \simeq c_{x_0}$
	$\bar{h} \cdot h \simeq c_{x_1}$ $\quad (*)$

Define a change of basepoint map

$$\beta_h : \pi_1(X, x_1) \longrightarrow \pi_1(X, x_0)$$

$$\beta_h([f]) = [\underbrace{h \cdot f \cdot \bar{h}}_{n\text{-fold prod.}}]$$

<u>well-defined</u> :
$f \simeq f'$ in $\Omega(X, x_1)$ $\Longrightarrow$ $h \cdot f \cdot \bar{h} \simeq h \cdot f' \cdot \bar{h}$ (thanks to Fact 0)

Prop. The map $\beta_h : \pi_1(X, x_1) \longrightarrow \pi_1(X, x_0)$ is an isomorphism.

Pf.: • β_h is a group homomorphism. why?

must check that $\beta_h([f]\cdot[g]) = \beta_h([f])\cdot\beta_h([g])$

but by def. $\beta_h([f]\cdot[g]) = \beta_h([f\cdot g])$

$$= [h\cdot(f\cdot g)\cdot\bar{h}]$$

now:

$$h\cdot(f\cdot g)\cdot\bar{h} \overset{\text{Fact 1}}{\simeq} h\cdot f\cdot g\cdot\bar{h}$$

Fact 0 $\simeq h\cdot(f\cdot c_{x_1})\cdot g\cdot\bar{h}$

Fact 1 $\simeq h\cdot f\cdot c_{x_1}\cdot g\cdot\bar{h}$

(*) ⊕ Fact 0 $\simeq h\cdot f\cdot(\bar{h}\cdot h)\cdot g\cdot\bar{h}$

Fact 1 $\simeq h\cdot f\cdot\bar{h}\cdot h\cdot g\cdot\bar{h}$

Fact 1 $\simeq (h\cdot f\cdot\bar{h})\cdot(h\cdot g\cdot\bar{h})$

recall that

$$\beta_h([f])\cdot\beta_h([g]) = [h\cdot g\cdot\bar{h}]\cdot[h\cdot g\cdot\bar{h}]$$

$$= [(h\cdot g\cdot\bar{h})\cdot(h\cdot g\cdot\bar{h})]$$

$$= \beta_h([f]\cdot[g])$$

• β_h is a bijection w/ inverse $\beta_{\bar{h}}$ defined by $\beta_{\bar{h}}([f]) = [\bar{h}\cdot f\cdot h]$

<u>check</u>:
$$\begin{cases} \beta_{\bar{h}} \circ \beta_h = \mathrm{id}_{\pi_1(X, x_0)} \\ \beta_h \circ \beta_{\bar{h}} = \mathrm{id}_{\pi_1(X, x_1)} \end{cases} \qquad \square$$

Moral of the story: when X is path-connected we only write $\pi_1(X)$ instead of $\pi_1(X, x_0)$ although a) the isomorphism is not canonical

b) $\pi_1(X)$ can be interpreted as (equivalence classes of) loops only in presence of a basepoint.

Fundamental group as a functor

(X, x_0) pointed space

(Y, y_0) " "

Let $f: (X, x_0) \to (Y, y_0)$ be a morphism of pointed spaces i.e. a continuous map $f: X \to Y$ w/ $f(x_0) = y_0$. This map induces

$f_*: \Omega(X, x_0) \longrightarrow \Omega(Y, y_0)$

by $\gamma \longmapsto f \circ \gamma$

We have seen (in L15) that

$\gamma \simeq \gamma' \implies f \circ \gamma \simeq f \circ \gamma'$

hence we have a well-defined map in

the quotient $f_*: \pi_1(X, x_0) \longrightarrow \pi_1(Y, y_0)$

$$[\gamma] \longmapsto [f\circ\gamma]$$

<u>Prop.</u> this map is a group homomorphism.

<u>Pf.</u> $f_*([\alpha]\cdot[\beta]) = f_*([\alpha\cdot\beta])$

$= [f\circ(\alpha\cdot\beta)]$ but (set-theoretic id.)

$f\circ(\alpha\cdot\beta) \equiv (f\circ\alpha)\cdot(f\circ\beta)$

$= [f\circ\alpha]\cdot[f\circ\beta] = f_*([\alpha])\cdot f_*([\beta])$ □

<u>Reality checks:</u>

F1) $(\mathrm{id}_X)_* = \mathrm{id}_{\pi_1(X)}$

F2) $(g\circ f)_* = g_* \circ f_*$

<u>Cor.</u> If $(X, x_0) \underset{g}{\overset{f}{\rightleftarrows}} (Y, y_0)$ are <u>homeomorphisms</u>, then the induced map $\pi_1(X, x_0) \underset{g_*}{\overset{f_*}{\rightleftarrows}} \pi_1(Y, y_0)$ are <u>isomorphisms</u>.

<u>Pf.</u> $(g\circ f)_* \overset{F2}{=} g_*\circ f_*$ | $f_*\circ g_*$

$= (\mathrm{id}_X)_* \overset{F1}{=} \mathrm{id}_{\pi_1(X)}$ | $= \mathrm{id}_{\pi_1(Y)}$

□