Prop.: Let $X$, $Y$ be top. spaces, and let $f, g : X \to Y$ be homotopically equivalent. Given any $x_0 \in X$ there is an isomorphism $\psi : \pi_1(Y, f(x_0)) \to \pi_1(Y, g(x_0))$ that makes the following diagram commute:

\[
\begin{array}{c}
\pi_1(X, x_0) \\
\bigg\downarrow g_* \\
\pi_1(Y, g(x_0))
\end{array}
\quad \quad \quad
\begin{array}{c}
\pi_1(Y, f(x_0)) \\
\downarrow f_*
\end{array}
\quad \quad \quad
\begin{array}{c}
\pi_1(Y, g(x_0))
\end{array}
\]

i.e. $g_* = \psi \circ f_*$. 

Cor. An homotopic equivalence $f : X \to Y$ induces an isomorphism at the level of fundamental groups, namely $\forall x_0 \in X$ the map $f_* : \pi_1(X, x_0) \to \pi_1(Y, f(x_0))$ is an isomorphism.

Hence: path-connected spaces that are homotopically equivalent have isomorphic fundamental groups.

\[
\begin{array}{c}
(X \cong Y) \quad \Rightarrow \quad \pi_1(X) \cong \pi_1(Y)
\end{array}
\quad \quad \quad
\oplus \quad \text{path-connected}
\]
Example: contractible spaces have trivial fund. group (hence, by def., they are simply connected) (rule: contractible $\Rightarrow$ path-connected, ex 8.1).

Last time we saw that if $K \subseteq \mathbb{R}^n$ is convex then it simply connected (e.g. $K = \mathbb{R}^n, B^4, D^4$).

Now we have many more examples of simply connected spaces, like:

a) star-shaped domains
b) snake-like domains

Proof of Corollary (given the Prop.)

By def. (of homotopic equivalence)

$g: y \rightarrow x$ such that

$\quad g \circ f \cong \text{id}_X$, $f \circ g \cong \text{id}_Y$

Hence (by prop.) $(g \circ f)_* = \psi \circ (\text{id}_X)_*$

$\cong \psi$
⇒ \( g* \circ f* = \psi \) \( \Rightarrow f* \) is injective

Similarly, we get that

\[
f* \circ g* : \pi_1(X, x_0) \to \pi_1(Y, f(g(x_0)))
\]

is an isomorphism \( \Rightarrow f* \) is surjective

Conclusion: \( f* \) is a bijective group hom. i.e. \( f* \) is an isomorphism. \( \square \)

**Proof.** (proposition)

- (warm-up case) Assume \( f(x_0) = g(x_0) \) and \( f \circ g \) through a homotopy preserving the basepoint i.e. \( \exists H : X \times I \to Y \) w/ \( H(x, 0) = f(x) \), \( H(x, t) = g(x) \) and \( H(x_0, t) = f(x_0) \) \( \forall t \in I \).

If so, can take \( \psi = \text{id} : \pi_1(Y, f(x_0)) \to \pi_1(X, x_0) \) for indeed

\[
f* (\gamma^{-1}) = g* (\gamma^{-1})
\]

\( \forall [\gamma^{-1}] \in \pi_1(X, x_0) \)

\( \Rightarrow f \circ f = g \circ f \) so loops in \( Y \) based at \( f(x_0) = g(x_0) \) and indeed the homotopy connecting these loops is just \( K : I \times I \to Y \) given by

\[
K(s, t) = H(t(s), t)
\]
(general case: must take care of basepoints)

Let $H : X \times I \rightarrow Y$ be a homotopy connecting $f$ to $g$, i.e. $H(x, 0) = f(x)$, $H(x, 1) = g(x)$ (but no extra conditions!) Fix $x_0 \in X$ so that statement, set $h(t) := H(x_0, t) \quad \exists$ change of basepoint map

$\beta \overline{h} : \pi_1(Y, f(x_0)) \rightarrow \pi_1(Y, g(x_0))$

$\gamma \quad \pi_1(Y, f(x_0)) \quad \pi_1(Y, g(x_0))$

Claim: $\gamma = \beta \overline{h}$ makes the diagram commute i.e. $g_* = \beta \overline{h} \circ f_*$. Let $\sigma \in \Omega(X, x_0)$, we have to prove $g \circ \overline{h} \sim \overline{h} \cdot (f \circ \sigma) \cdot h$ as loops in $\Omega(Y, g(x_0))$. 
Let \( \overline{h}_t : [0, 1] \to Y \) be the reparametrization of \( h \) restricted to \( [0, t] \), let \( h_0 \) be the inverse of \( h \)

\[
\overline{h}_t (s) = h(t(s)) = h(1-ts).
\]

Then define the loops

\[
\alpha_t (s) = H(\overline{h}_t(s), t)
\]

Note that

\[
\begin{align*}
\alpha_0 (s) &= H(\overline{h}_0(s), 1) = g \circ f^{-1} (s) \\
\alpha_1 (s) &= H(\overline{h}_1(s), 0) = f \circ g^{-1} (s)
\end{align*}
\]

Set \( K : I \times I \to Y \) by declaring

\[
K(s, t) = \overline{h}_t \cdot \alpha_t \cdot h(t(s))
\]

Check that

\[
\begin{align*}
K(s, 0) &= \overline{h}_0 \cdot (g \circ f^{-1}) \cdot h_0 = g \circ f^{-1} \\
K(s, 1) &= \overline{h}_1 \cdot (f \circ g^{-1}) \cdot h_1 \\
&= h \cdot (f \circ g^{-1}) \cdot h
\end{align*}
\]

\( K \) is an homotopy \( \overline{h} \sim (g \circ f, g(x_0)) \) connecting \( \overline{h} \cdot (f \circ g^{-1}) \cdot h \) and \( \overline{h}_0 \cdot (g \circ f^{-1}) \cdot h_0 \), but \( \overline{h}_0 \cdot (g \circ f^{-1}) \cdot h_0 \sim g \circ f^{-1} \Rightarrow g_k = f_k. \)