Homotopic Invariance of Fundamental Group - Lecture 17

Prop.: Let X, Y be top. spaces, and let $f, g : X \to Y$ be homotopically equivalent. Given any $x_0 \in X$ there is an isomorphism $\psi : \pi_1(Y, f(x_0)) \longrightarrow \pi_1(Y, g(x_0))$ that makes the following diagram commute:

$$
\begin{tikzcd}
 & \pi_1(Y, f(x_0)) \arrow[d, dashed, "\psi"] \\
\pi_1(X, x_0) \arrow[ur, "f_*"] \arrow[r, "g_*"'] & \pi_1(Y, g(x_0))
\end{tikzcd}
$$

i.e. $g_* = \psi \circ f_*$.

Cor. An homotopic equivalence $f : X \longrightarrow Y$ induces an isomorphism at the level of fund. groups, namely $\forall x_0 \in X$ the map

$$f_* : \pi_1(X, x_0) \longrightarrow \pi_1(Y, f(x_0))$$

is an isomorphism.

Hence: path-connected spaces that are homotopically equivalent have isomorphic fund. groups

$$\left(X \simeq Y \Longrightarrow \pi_1(X) \simeq \pi_1(Y) \right)$$

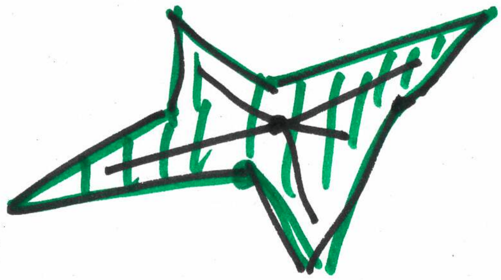
$\oplus$ path-connected

<u>Example</u>: contractible spaces have trivial fund. group (hence, by def., they are simply connected) (<u>rmk.</u> contractible $\Rightarrow$ path-connected, ex 8.1).

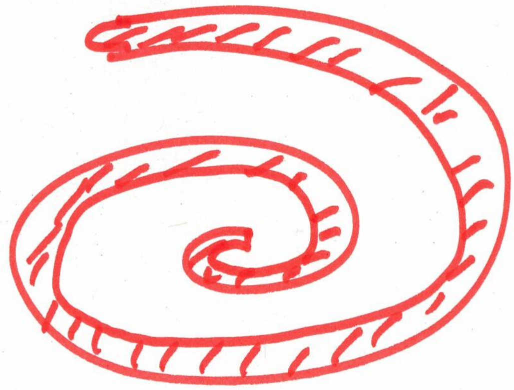
Last time we saw that if $K \subseteq \mathbb{R}^n$ is <u>convex</u> then it simply connected (e.g. $K = \mathbb{R}^n, B^n, D^n$).

Now we have many more examples of simply connected spaces, like

a) star-shaped domains



b) snake-like domains



<u>Proof of Corollary</u> (given the Prop.)

By def. (of homotopic equivalence)

$\exists\ g : Y \longrightarrow X$ such that

$$g \circ f \simeq id_X \ , \quad f \circ g \simeq id_Y$$

Hence (by prop.) $(g \circ f)_* = \varphi \cdot (id_X)_*$

$$= \varphi$$

$\Longrightarrow \quad g_* \circ f_* = \psi \quad \Longrightarrow \quad f_*$ is <u>injective</u>

Similarly, we get that

$$f_* \circ g_* : \pi_1(X, x_0) \longrightarrow \pi_1(Y, f(g(x_0)))$$

is an isomorphism $\Longrightarrow$ f_* is <u>surjective</u>

<u>Conclusion</u>: f_* is a bijective group hom.
i.e. f_* is an isomorphism. □

<u>Proof.</u> (proposition)

- (warm-up case) assume $f(x_0) = g(x_0)$ and $f \simeq g$ through an homotopy preserving the basepoint i.e. $\exists\ H : X \times I \longrightarrow Y$ w/ $H(x, 0) = f(x)$, $H(x, 1) = g(x)$ and $H(x_0, t) = f(x_0)\ \forall t \in I$.

If so, can take $\psi = \mathrm{id} : \pi_1(Y, f(x_0))$
for indeed $\boxed{f_*([\sigma]) = g_*([\sigma])}$
$\forall [\sigma] \in \pi_1(X, x_0)$

$\Longleftrightarrow$ $f \circ \sigma \simeq g \circ \sigma$ as loops in Y, based at $f(x_0) = g(x_0)$

and indeed the homotopy connecting these loops is just $K : I \times I \longrightarrow Y$ given by

$$K(s, t) = H(\sigma(s), t)$$

• (general case: must take care of basepoints)

Let $H : X \times I \longrightarrow Y$ be an homotopy connecting f to g, i.e. $H(x,0) = f(x)$, $H(x,1) = g(x)$ (but no extra conditions!) Fix $x_0 \in X$ as in the statement, set $h(t) := H(x_0, t)$

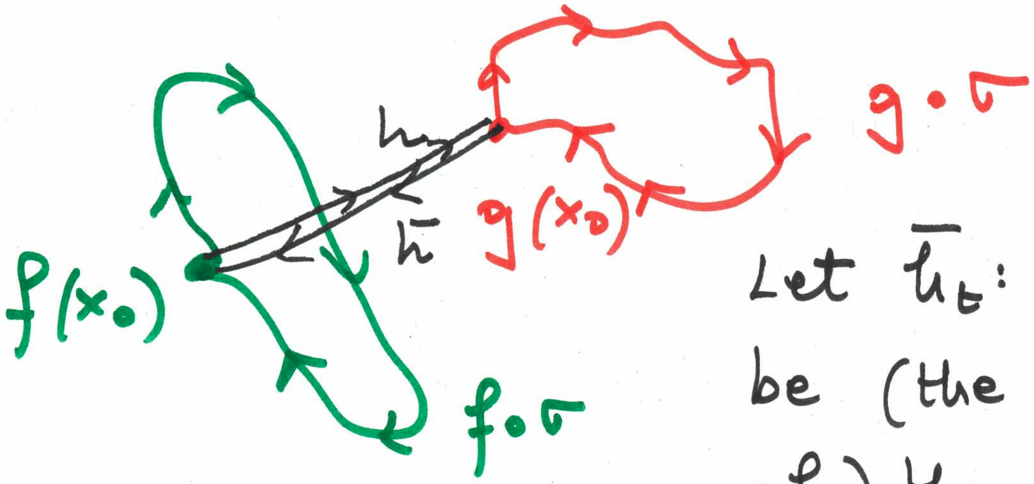
$\exists$ <u>change of basepoint map</u> L 16

$$\beta_{\bar h} : \pi_1(Y, f(x_0)) \longrightarrow \pi_1(Y, g(x_0))$$
$$[\alpha] \longmapsto [\bar h \cdot \alpha \cdot h]$$

which we proved (last time) to be an <u>isomorphism</u>.

<u>Claim</u>: $\psi = \beta_{\bar h}$ makes the diagram commute i.e. $g_* = \beta_{\bar h} \circ f_*$.

Let $\sigma \in \Omega(X, x_0)$, we have to prove $g \circ \sigma \simeq \bar h \cdot (f \circ \sigma) \cdot h$ as loops in $\Omega(Y, g(x_0))$.



Let $\bar{h}_t : [0,1] \to Y$ be (the reparametrization of) the restriction of $\bar{h}$ to $[0,t]$, let h_t be the inverse of $\bar{h}_t$ i.e. $\bar{h}_t(s) = \bar{h}(ts) = h(1-ts)$. Then define the loops $\alpha_t(s) = H(\sigma(s), 1-t)$

note that
$$\begin{cases} \alpha_0(s) = H(\sigma(s), 1) = g \circ \sigma(s) \\ \alpha_1(s) = H(\sigma(s), 0) = f \circ \sigma(s) \end{cases}$$

Set $K : I \times I \longrightarrow Y$ by declaring
$$K(s,t) = \bar{h}_t \cdot \alpha_t \cdot h_t(s)$$

Check that

$\boxed{t=0}$ $\quad K(s,0) = \bar{h}_0 \cdot (g \circ \sigma) \cdot h_0 \simeq g \circ \sigma$

$\boxed{t=1}$ $\quad K(s,1) = \bar{h}_1 \cdot (f \circ \sigma) \cdot h_1$
$$\equiv \bar{h} \cdot (f \circ \sigma) \cdot h$$

K is an homotopy in $\Omega(Y, g(x_0))$ connecting $\bar{h} \cdot (f \circ \sigma) \cdot h$ and $\bar{h}_0 \cdot (g \circ \sigma) \cdot h_0$ but $\bar{h}_0 \cdot (g \circ \sigma) \cdot h_0 \simeq g \circ \sigma \Rightarrow g_* = \psi f_*$. $\square$