Covering Spaces - Lecture 18

Def. Let $X, \tilde{X}$ be top. spaces and let $p: \tilde{X} \to X$.
We say that $U \subset X$ open is <u>evenly covered</u> if

$$p^{-1}(U) = \coprod_{i \in I} U_i \quad \text{(disjoint union)}$$

and $p|_{U_i} : U_i \to U$ homeomorphism.

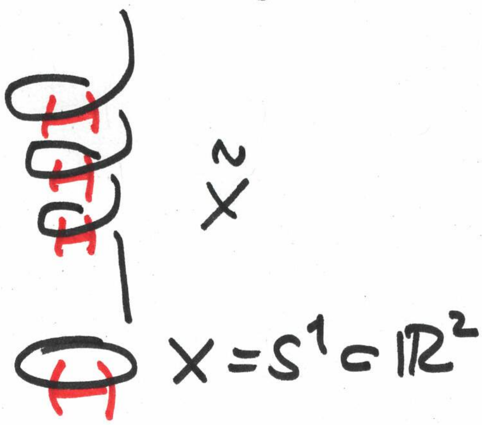
Def. Let $X, \tilde{X}$ be top. spaces and let $p: \tilde{X} \to X$.
We say that p is a <u>covering map</u> if $\forall x \in X$ $\exists$ $U = U(x)$ open neighborhood of x that is evenly covered.

Examples: a) $X = S^1$, $\tilde{X} = \mathbb{R}$ $p: \tilde{X} \to X$

$$p(s) = (\cos(2\pi s), \sin(2\pi s)).$$

<u>Claim</u> (exercise 9.1): this is a covering map.

Visualization: embed $\tilde{X} = \mathbb{R}$ in $\mathbb{R}^3$ as the helix $\varphi(s) = (\cos(2\pi s), \sin(2\pi s), s)$
consider the projection $\pi : \mathbb{R}^3 \equiv \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}^2$ given by $\pi(x, y, z) = (x, y)$. Then $p = \pi \circ \varphi$



note that $\forall x \in S^1$ $p^{-1}(x)$ has infinite cardinality.

b) $X = S^1$, $\tilde{X} = S^1$ $p: \tilde{X} \to X$
$p(z) = z^n$ $(S^1 \subset \mathbb{C})$

<u>Claim (exercise 9.2)</u>: this is a covering map.

$\forall x \in X$ have $|p^{-1}(x)| = n$ $(n \geq 1)$.

<u>Def.</u> Let X, Y be top. spaces and $f: X \to Y$. Then $f^{-1}(y)$ is called <u>fiber</u> of y.

<u>Prop.</u> Let X, Y be top. spaces, Y connected. Assume $f: X \to Y$, then:

(i) If f is a covering map then it is a <u>local homeomorphism</u> and the cardinality of $f^{-1}(y)$ does not depend on y. [cardinality $\in \overline{\mathbb{N}} = \mathbb{N} \cup \{\infty\}$]

(ii) If f is a <u>local homeomorphism</u> and the fibers $f^{-1}(y)$ have <u>finite</u> cardinality (whose value does not depend on y) then f is a covering map.

Recall: $f: X \to Y$ is a local homeomorphism if $\forall x \in X$ $\exists$ $\begin{cases} U = U(x) \text{ open} \subset X \\ V = V(f(x)) \text{ open} \subset Y \end{cases}$

w/ $f|_U : U \to V$ is a homeomorphism.

<u>Proof.</u> (i) f covering $\Rightarrow$ f local homeo is trivial. Let's check constancy of $|f^{-1}(y)|$. Equiv. rel:

$y_1 \sim y_2 \iff |f^{-1}(y_1)| = |f^{-1}(y_2)|$ in $\overline{\mathbb{N}}$

Say $Y(n) \subset Y$ the set of points w/ $n \in \overline{\mathbb{N}}$ elements in the fiber.

$Y = \coprod_{n \in \overline{\mathbb{N}}} Y(n)$ but $\forall n \in \overline{\mathbb{N}}$ have that $Y(n)$ is <u>open</u>. Hence:

Y connected $\Rightarrow \exists n_0 \in \overline{\mathbb{N}}$ w/ $Y = Y(n_0)$.

(ii) Given $y \in Y$, let $f^{-1}(y) = \{x_1, \dots, x_k\}$ so let's build an evenly covered neighb. of y.
By def of local homeo $\forall i \in \{1, \dots, k\}$ $\exists U_i \subset X$ open w/ $f|_{U_i}: U_i \longrightarrow f(U_i)$ homeo. Take

$$\bigcap_{i \in \{1, \dots, k\}} f(U_i) = V$$

$\leftarrow$ open set in Y

<u>claim</u>: evenly covered

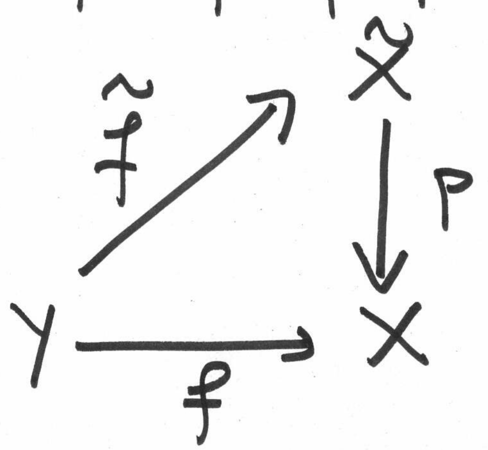
Set $U_i' = U_i \cap f^{-1}(V)$ open in X

Clearly $f^{-1}(V) \supset U_i'$ $\forall i$ $\Rightarrow f^{-1}(V) \supset \bigcup_i U_i'$
but any point of Y has exactly k preimages (not more!) so in fact $\boxed{f^{-1}(V) = \bigcup_i U_i'}$

($\forall v \in V$ have $f^{-1}(v) \cap U_i' \neq \phi$, contains one element). Then, by restriction, since $U_i' \subset U_i$ have that $f|_{U_i'}: U_i' \to f(U_i')$ is a homeomorphism. $\square$

<u>Rmk</u>. to check that $f: U \to V$ for $U, V \subset \mathbb{R}^n$ open is a local homeo is easy because it's enough (if $f \in C^1$) $\det(\mathrm{Jac}(f(x_0))) \neq 0$ ($\Rightarrow$ local diffeo)

Lift(s): Let $p : \tilde{X} \to X$ be a covering map and let $f : Y \to X$, $\tilde{f} : Y \to \tilde{X}$ (continuous).
If $f = p \circ \tilde{f}$ we say that $\tilde{f}$ is a lift of f.



- q: given f is there always a lift $\tilde{f}$? unique?

Thm (uniqueness, Prop. 1.34) Let Y be connected.
If $\tilde{f}_1, \tilde{f}_2 : Y \to \tilde{X}$ are both lifts of $f : Y \to X$ and $\tilde{f}_1(y_0) = \tilde{f}_2(y_0)$ for some $y_0 \in Y$ then $\tilde{f}_1 = \tilde{f}_2$.

Pf. $S = \{ y \in Y : \tilde{f}_1(y) = \tilde{f}_2(y) \}$

claim: ↑ is open and closed in Y ($\underset{Y \text{ conn.}}{\Longrightarrow} S = Y$).

For $y \in Y$ take $f(y) \in U$ open evenly covered so $p^{-1}(U) = \bigcup_{i \in I} U_i$ w/ $p|_{U_i} : U_i \xrightarrow[\text{homeo}]{} U$.

$\exists$ (by continuity) open neigh. $N = N(y)$ of $y \in Y$
w/ if $\tilde{f}_1(y) \in U_1 \Rightarrow \tilde{f}_1(N) \subset U_1$
if $\tilde{f}_2(y) \in U_2 \Rightarrow \tilde{f}_2(N) \subset U_2$

- if $\tilde{f}_1(y) \neq \tilde{f}_2(y) \Rightarrow U_1 \neq U_2$ (i.e. $U_1 \cap U_2 = \emptyset$)
  so $N(y) \subset Y \setminus S \Rightarrow Y \setminus S$ open $\Leftrightarrow$ S closed
- if $\tilde{f}_1(y) = \tilde{f}_2(y) \Rightarrow U_1 = U_2 =: \tilde{U}$ and $p|_{\tilde{U}} : \tilde{U} \to U$ bijection so on $N(y)$ $\tilde{f}_1 = (p|_{\tilde{U}})^{-1} \circ f = \tilde{f}_2$.

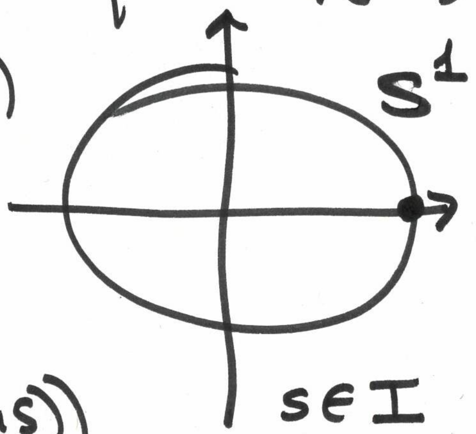
<u>Existence:</u> Let $p: \tilde{X} \to X$ be a coveringmap.

(a) given a path $\alpha: I \to X$ and $\tilde{x}_0 \in p^{-1}(\alpha(0))$
$\exists$! lift $\tilde{\alpha}: I \to \tilde{X}$ of α, w $\tilde{\alpha}(0) = \tilde{x}_0$.

(b) given $H: I \times I \to X$ homotopy of loops based at x_0, and $\tilde{x}_0 \in p^{-1}(x_0)$
$\exists$! lift $\tilde{H}: I \times I \to \tilde{X}$ of H, that is a homotopy of paths starting at $\tilde{x}_0 \in \tilde{X}$.

Giving this for granted, let's compute $\pi_1(S^1)$.

$S^1 \subset \mathbb{C}$ unit circle, $x_0 = (1,0)$



$p: \mathbb{R} \to S^1$

$p(s) = (\cos(2\pi s), \sin(2\pi s))$

$\omega_n(s) = (\cos(2\pi n s), \sin(2\pi n s))$ $\quad s \in I$

$\tilde{\omega}_n(s) = ns \qquad (n \in \mathbb{Z}) \qquad s \in I$

<u>check:</u> $\tilde{\omega}_n$ is the only lift of ω_n such that $\tilde{\omega}_n(0) = 0 \in \mathbb{R}$.

<u>Thm:</u> $\pi_1(S^1, (1,0)) \cong \mathbb{Z}$, in fact it is the infinite cyclic group generated by $[\omega_1]$.

<u>Proof.</u>

$$\mathbb{Z} \xrightarrow{\Phi} \pi_1(S^1, x_0)$$
$$n \longmapsto [\omega_n]$$

- group hom: $\Phi(n+m) = [\omega_{n+m}]$

$= [\omega_n \cdot \omega_m] = [\omega_n] \cdot [\omega_m] = \Phi(n) \cdot \Phi(m)$

- <u>surjective</u>: given $\alpha \in \Omega(S^1, x_0)$
let $\tilde{\alpha}$ be the only lift of α starting at the origin $0 \in \mathbb{R}$. Key remark:
$\tilde{\alpha}(1) \in \mathbb{Z} \subset \mathbb{R}$ i.e. $\tilde{\alpha}(1) = n$
for some $n \in \mathbb{Z}$ (because $p^{-1}(x_0) = \mathbb{Z}$)
hence we claim that $\tilde{\alpha} \simeq \tilde{\omega}_n$, in fact we can take a linear homotopy

$$\tilde{F}(s,t) = (1-t)\,\tilde{\alpha}(s) + t\,\tilde{\omega}_n(s)$$

$\Rightarrow F = p \circ \tilde{F}$ is a homotopy in $\Omega(S^1, x_0)$

w/ $F(s,0) = \alpha$ and $F(s,1) = \omega_n$ i.e.
$\Phi(n) = [\alpha]$.

- <u>Injective</u>: we must show

$$\omega_n \simeq \omega_m \quad \Longrightarrow \quad m = n$$

as loops in $\Omega(S^1, x_0)$
If F is this homotopy, lift it to $\tilde{F}: I \times I \to \mathbb{R}$ (of paths starting at $0 \in \mathbb{R}$)

Have: $\begin{cases} \tilde{F}(s,0) = \tilde{\omega}_n(s) \\ \tilde{F}(s,1) = \tilde{\omega}_m(s) \\ \tilde{F}(0,t) = 0 \end{cases}$ but since I'm lifting <u>loops</u> have $\tilde{F}(1,t) \in \mathbb{Z}$ $\forall t \in I$

by connectedness $\tilde{F}(1,t) = \text{constant (w.r.t. } t)$

$$\Rightarrow \tilde{F}(1,0) = \tilde{F}(1,1)$$

$$\tilde{F}(1,0) = \tilde{\omega}_n(1) = n, \quad \tilde{F}(1,1) = \tilde{\omega}_m(1) = m \quad \Longrightarrow n = m \quad \square$$

<u>Fundamental group of products</u>

recall (universal property of products):

$f: Z \longrightarrow X \times Y$ is continuous iff its components $\underbrace{\pi_X \circ f}_{f_X}$, $\underbrace{\pi_Y \circ f}_{f_Y}$ are.

In part:

a) if $\alpha: I \longrightarrow X \times Y$ is a loop based at (x_0, y_0) then $\alpha = (\alpha_x, \alpha_y)$ w/ $\alpha_x \in \Omega(X, x_0)$ and $\alpha_y \in \Omega(Y, y_0)$.

b) if $F: I \times I \longrightarrow X \times Y$ is a homotopy of loops in $\Omega(X \times Y, (x_0, y_0))$ then $F = (F_x, F_y)$ w/ F_x homotopy of loops in $\Omega(X, x_0)$, same for F_y

Thm. Given X, Y path-connected top. spaces, $x_0 \in X$, $y_0 \in Y$ have

$$\pi_1(X \times Y, (x_0, y_0)) \cong \pi_1(X, x_0) \times \pi_1(Y, y_0)$$

Cor. $\pi_1(S^1 \times S^1) \cong \mathbb{Z} \times \mathbb{Z}$

((p,q) is a loop that winds around p times in the first factor and q times in second factor)

Proof. Set $\Psi : \pi_1(X \times Y, (x_0, y_0)) \longrightarrow \pi_1(X, x_0) \times \pi_1(Y, y_0)$

$$\Psi([\alpha]) = ([\alpha_X], [\alpha_Y])$$

- well-defined by b)
- group homomorphism

$$\Psi([\alpha] \cdot [\beta]) = \Psi([\alpha \cdot \beta])$$

$$= ([(\alpha \cdot \beta)_X], [(\alpha \cdot \beta)_Y])$$

$$= ([\alpha_X] \cdot [\beta_X], [\alpha_Y] \cdot [\beta_Y])$$

$$= ([\alpha_X], [\alpha_Y]) \times ([\beta_X], [\beta_Y])$$

$$= \Psi([\alpha]) \times \Psi([\beta])$$

- injective by b)
- surjective $([\sigma], [\tau]) \in \pi_1(X, x_0) \times \pi_1(Y, y_0)$ then $\alpha := (\sigma, \tau)$ have $[\alpha] \in \pi_1(X \times Y, (x_0, y_0))$ and $\Psi([\alpha]) = ([\sigma], [\tau])$. □