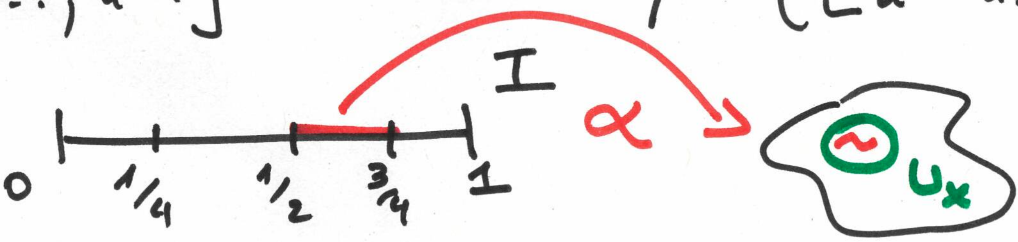
Existence of Lift(s) - Lecture 19

1) lifting paths:

Prop. Let $p: \tilde{X} \to X$ be a covering map.
Given a path $\alpha: I \to X$ and $\tilde{x}_0 \in p^{-1}(\alpha(0))$
$\exists!$ lift $\tilde{\alpha}: I \to \tilde{X}$ of α w/ $\tilde{\alpha}(0) = \tilde{x}_0$.

Lemma: Given $p: \tilde{X} \to X$ a covering map and $\alpha: I \to X$, for $x \in X$ let U_x be an open neighborhood of x (in part. $\bigcup_x U_x = X$).
There exists $n \in \mathbb{N}$ such that
$\forall i \in \{0, \dots, n-1\}$ $\exists x \in X$ w/ $\alpha([\frac{i}{n}, \frac{i+1}{n}]) \subset U_x$.

picture:



Proof (Lemma)

Consider the open cover of $I = [0,1]$ given by $\{\alpha^{-1}(U_x)\}_{x \in X}$. By compactness of this cover has a positive Lebesgue number i.e.
$\exists\ \epsilon > 0$ s.t. $\forall s \in I$ $\exists x \in X$ w/

$$\underset{(s-\epsilon, s+\epsilon) \cap I}{\underset{|||}{B_\epsilon(s)}} \subset \alpha^{-1}(U_x)$$

. Then pick $n \in \mathbb{N}$ w/ $\frac{1}{n} < \epsilon$

$B_{1/n}(s) \subset B_\epsilon(s) \subset \alpha^{-1}(U_x)$ so apply α:

$\alpha(B_{1/n}(s)) \subset \alpha(B_\epsilon(s)) \subset \alpha(\alpha^{-1}(U_x)) = U_x$. Hence for that same n have $\left[\frac{i}{n}, \frac{i+1}{n}\right] \subset B_{1/n}(x_i)$

$\exists\, x \in X$ w/ $\alpha\left(\left[\frac{i}{n}, \frac{i+1}{n}\right]\right) \subset U_x$ □

<u>Proof</u> (prop.)

Given $\alpha : I \longrightarrow X$ as in the statement, consider $\{U_x\}_{x \in X}$ be a cover of X consisting of <u>evenly</u> covered open neighborhoods, take $n \in \mathbb{N}$ as prescribed by the lemma. Now, construct the lift "one subinterval at a time" starting w/ $I_0 = [0, \frac{1}{n}]$.

$\alpha(I_0)$ is contained in an evenly covered open set $U \subset X$ hence if $p^{-1}(U) = \coprod_{j \in J} U_j$

$\exists!\, j_* \in J$ w/ $\tilde{x}_0 \in U_{j_*}$. So define

$\tilde{\alpha}|_{I_0} : I_0 \longrightarrow \tilde{X}$ by $\tilde{\alpha} = \left(p|_{U_{j_*}}^{-1} \circ \alpha\right)$.

Then consider the endpoint $\tilde{\alpha}(1/n)$ and proceed inductively (i.e. repeat the operation of lifting).

We lift $\alpha|_{[1/n, 2/n]}$ starting at $\tilde{\alpha}(1/n) \in \tilde{X}$.

After exactly n steps we get $\tilde{\alpha} : I \longrightarrow \tilde{X}$ lifting α, and starting at given $\tilde{x}_0$. □

2) Lift homotopies:

<u>Prop.</u>: Let $p: \tilde{X} \longrightarrow X$ be a covering map. Given $H: I \times I \longrightarrow X$ homotopy of loops based at x_0 and $\tilde{x}_0 \in p^{-1}(x_0)$ $\exists !$ lift $\tilde{H}: I \times I \longrightarrow \tilde{X}$ homotopy of paths (all) starting at $\tilde{x}_0 \in \tilde{X}$.

<u>Lemma</u>: (pb. 9.9) In the setting above, given an open cover $\{U_x\}_{x \in X}$ of X there exists $n \in \mathbb{N}$ s.t. $\forall\, i,j \in \{0, \dots, n-1\}$ $\exists\, x \in X$

$$H\Big(\underbrace{\left[\tfrac{i}{n}, \tfrac{i+1}{n}\right] \times \left[\tfrac{j}{n}, \tfrac{j+1}{n}\right]}_{Q_{i,j}}\Big) \subset U_x .$$

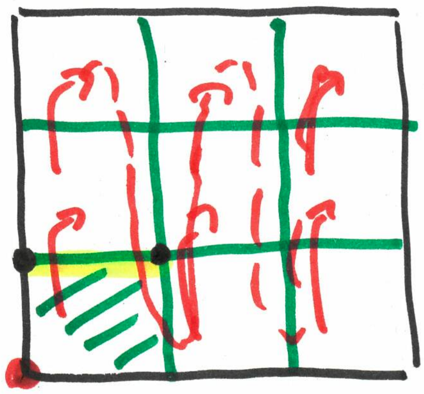
<u>Proof.</u> (prop.) We construct the lift "one subsquare at a time", starting from $Q_{0,0}$:

$\tilde{H}\big|_{Q_{0,0}} : Q_{0,0} \longrightarrow \tilde{X}$ defined by

$$\tilde{H} = \left(p\big|_{U_{j^*}}^{-1} \circ H\right)$$

where $H(Q_{0,0}) \subset U$, $p^{-1}(U) = \coprod_{j \in J} U_j$ and $\tilde{x}_0 \in U_{j^*}$. We proceed inductively following the order

$Q_{0,0}, Q_{0,1}, \dots, Q_{0,n-1}, Q_{1,0}, \dots$



To define $\tilde{H}|_{Q_{0,1}}$:

- look at $H(Q_{0,1})$

$\overset{\text{lemma}}{\rightsquigarrow} \exists\ U' \subset X$ evenly covered w/ $H(Q_{0,1}) \subset U'$

- $p^{-1}(U') = \coprod_{j \in J'} U'_j$, $\exists!\ j'_* \in J'$

w/ $\tilde{H}|_{Q_{0,0}}(0, 1/n) \in U'_{j'_*}$

- define $\tilde{H}|_{Q_{0,1}} = \left(p|_{U'_{j'_*}}^{-1} \circ H \right)$

?: why do $\tilde{H}|_{Q_{0,0}}$ and $\tilde{H}|_{Q_{0,1}}$ coincide along the segment $L := \{1/n\} \times [0, 1/n] \subset \mathbb{R}^2$?

It's enough to note that $\tilde{H}|_{Q_{0,0}}$ and $\tilde{H}|_{Q_{0,1}}$ when restricted to L are two lifts of the same path (namely $H|_L$) starting at the same point. Hence (by uniqueness) the two definitions agree at the interface $\Rightarrow$ after n^2 steps get a well-defined $\tilde{H}: I \times I \to \tilde{X}$ w/ desired properties. $\square$

<u>Cor</u> (Monodromy Thm) Let $p: \tilde{X} \to X$ be a covering map and let $\tilde{\alpha}, \tilde{\beta}: I \to \tilde{X}$ be paths. Assume that $\alpha := p \circ \tilde{\alpha}$, $\beta = p \circ \tilde{\beta}$ are homotopic w/ fixed endpoints in X. Then:

$$\tilde{\alpha}(0) = \tilde{\beta}(0) \iff \tilde{\alpha}(1) = \tilde{\beta}(1).$$

<u>Proof.</u> ($\Rightarrow$) Let $F: I \times I \to X$ be a homotopy w/ $F(s,0) = \alpha(s)$, $F(s,1) = \beta(s)$ and $F(0,t) = x_0$, $F(1,t) = x_1$.

If $\tilde{\alpha}(0) = \tilde{\beta}(0) =: \tilde{x}_0$ lift F to $\tilde{F}$ starting at $\tilde{x}_0$. Now $\underbrace{\tilde{F}(0,t) \in p^{-1}(x_0)}$ $(\forall t \in I)$

this fiber is a discrete subset of $\tilde{X}$ (cf. problem 7.7)

hence by connectedness of $[0,1]$ have $\tilde{F}(0,t) = \text{constant} = \tilde{x}_0$. It follows that

$$\begin{cases} \tilde{F}|_{I \times \{0\}} = \tilde{\alpha} \\ \tilde{F}|_{I \times \{1\}} = \tilde{\beta} \end{cases}$$

(uniqueness of lifts of paths)

But $\tilde{F}(1,t)$ is also constant in t hence

$$\underset{\tilde{\alpha}(1)}{\underset{\|}{\tilde{F}(1,0)}} = \underset{\tilde{\beta}(1)}{\underset{\|}{\tilde{F}(1,1)}} \implies \tilde{\alpha}(1) = \tilde{\beta}(1)$$

$\square$