Van Kampen's Theorem – Lecture 20

<u>heuristic</u> (excision principle): if $X = A \cup B$ and we understand (i.e. we can compute π_1) of $A, B, A \cap B$ then we understand (i.e. we can compute π_1) of X

Free Product of Groups:

- Let S be a set ($S \neq \emptyset$). A word (in the alphabet S) is a finite sequence of letters in S, possibly empty. On the set of all words $W(S)$ there is a natural <u>product</u> (juxtaposition)

$$(p, p') \longmapsto pp'$$

i.e. if $\left.\begin{array}{l} p = s_1 \dots s_m \\ p' = s'_1 \dots s'_n \end{array}\right\} \Rightarrow pp' = s_1 \dots s_m s'_1 \dots s'_n$.

<u>Def.</u> The free product of two groups H and K is $H * K := W(H \amalg K)/\sim$ where $p \sim p'$ if p' can be obtained from p via a finite chain of moves which are:

M1) if $p = x_1 \dots x_m$ and $x_i, x_{i+1} \in H$ (or $\in K$) then I replace $x_i x_{i+1}$ by their product x in H (or K)

M2) if $p = x_1 \dots x_m$ and $x_i = e_H$ (or $x_i = e_K$) then I remove this letter from the string p;

or the inverse of these moves:

M1)' replace x by $x_i x_{i+1}$ (if $x_i x_{i+1} = x$ in H or in K)

M2)' insert identity element of either group.

<u>Prop.</u>: The product operation (as defined on $W(H \amalg K)$) descends to the quotient $H * K$ and defines a group.

<u>Proof.</u>: • if $p_1 \sim p_1'$, $p_2 \sim p_2'$ then $p_1 p_2 \sim p_1' p_2'$

because I just apply both sets of moves!

- associate is obvious
- identity: take the empty word
- inverse: if $p = a_1 \dots a_e$ set $p' = a_e^{-1} \dots a_1^{-1}$

then
$$
\begin{aligned}
p p' &= a_1 \dots a_e a_e^{-1} \dots a_1^{-1} \\
&\underset{(M1)}{\overset{\sim}{\cancel{=}}} a_1 \dots a_{e-1}\, e\, a_{e-1}^{-1} \dots a_1^{-1} \\
&\underset{(M2)}{\overset{\sim}{\cancel{=}}} a_1 \dots a_{e-1} a_{e-1}^{-1} \dots a_1^{-1} \overset{\sim}{\cancel{=}} \dots \overset{\sim}{\cancel{=}} \text{empty word}
\end{aligned}
$$
$\square$

<u>Prop.</u>: Every element except the empty word of $H * K$ can be written uniquely as a word $a_1 \dots a_e$ where

- $a_{2i+1} \in H$ and $a_{2i} \in K$ $\forall i$

  <u>or</u> viceversa $a_{2i} \in H$ and $a_{2i+1} \in K$ $\forall i$

- $a_i \neq e_H, e_K$ $\forall i$.

Such form is called reduced.

<u>Proof.</u> exercise (induction on the length of the given word).

<u>Cor. 1</u>: The map $\iota_H : H \longrightarrow H * K$ defined by mapping any $h \in H$ to the word with that <u>one</u> letter is an injective homomorphism (same story for $\iota_K : K \longrightarrow H * K$).

<u>Proof.</u> • to check that this is a homomorphism is equiv. to proving that if $h = h_1 h_2$ in H then $h \sim h_1 h_2$ (in $W(H \amalg K)$) true by M1) above.

• injective $\underbrace{\iota_H(h)}_{\text{is reduced}} = \text{empty word} \implies h = e_H$ in H □

<u>Cor. 2</u>: if H, K are both non-trivial groups then $H * K$ contains ∞ elements and is not abelian.

<u>Proof.</u> joke given the reduced form:

• $h \neq e_H$ in H reduced words h
$k \neq e_K$ in K
hk
hkh
$hkhk\ldots$

• $hk \neq kh$ again by uniqueness of reduced form.

Def. We define the <u>free group</u> (of rank two) on $\mathbb{Z} * \mathbb{Z}$ (i.e. the output of the construction above in the special case $H = K = \mathbb{Z}$).

Rmk. :

$\mathbb{Z} = \langle a \rangle$ 1st copy

$\mathbb{Z} = \langle b \rangle$ 2nd copy

a (reduced) word $a^{d_1} b^{d_2} a^{d_3} \dots b^{d_e}$

- Cor. 2 applies to this case: $\mathbb{Z} * \mathbb{Z}$ is not abelian.
- $Ab(\mathbb{Z} * \mathbb{Z}) = \mathbb{Z} \oplus \mathbb{Z}$ (abelian group of rank 2).

Fact: the whole discussion above can be extended to $\{G_\alpha\}_\alpha \longmapsto *_\alpha G_\alpha$

In particular one can define $\underbrace{\mathbb{Z} * \dots * \mathbb{Z}}_{\text{free group of rank } n}$.

Extension of group homomorphisms: given a collection of group hom.

$$\varphi_\alpha : G_\alpha \longrightarrow H \quad \text{(for some group } H)$$

"there is a unique group hom that extends them"

i.e. $\exists!\ \varphi : * G_\alpha \longrightarrow H$

w/ $\varphi_\alpha = \varphi \circ \iota_\alpha \ \forall \alpha$ where $\iota_\alpha : G_\alpha \to G$

Let's say you have a word $g_1 \cdots g_n$

product in H

$$\varphi(g_1 \cdots g_n) = \varphi(g_1) \cdot \ldots \cdot \varphi(g_n)$$

$$= \varphi_{\alpha_1}(g_1) \cdot \ldots \cdot \varphi_{\alpha_n}(g_n)$$

if $g_i \in G_{\alpha_i}$

Note that if $g_1 \cdots g_n \sim g_1' \cdots g_m'$ as words then by our def. $\varphi(g_1 \cdots g_n) = \varphi(g_1' \cdots g_m')$ because the image, through φ, of a word does <u>not</u> change when I perform <u>one</u> move.

(by explicit def. of M1, M2 and axioms/prop. of group hom).

Conclusion: φ is well-defined and (trivially) a group homomorphism $\varphi : * G_\alpha \to H$ extending the given ones.

Van Kampen: suppose $X = \bigcup_\alpha A_\alpha$

w/ A_α path-connected, open in X $(\forall \alpha)$

Suppose $\exists\, x_0 \in X$ w/ $x_0 \in \bigcap_\alpha A_\alpha$

basepoint for fundam group

There exist maps induced by the inclusions:

$$j_\alpha : \pi_1(A_\alpha) \longrightarrow \pi_1(X)$$

$$i_{\alpha\beta} : \pi_1(A_\alpha \cap A_\beta) \longrightarrow \pi_1(A_\alpha)$$

By previous discussion there is a group hom $\Phi : *_\alpha \pi_1(A_\alpha) \longrightarrow \pi_1(X)$ extending the single maps j_α.

Theorem: If X is the union of path-connected open sets each containing the basepoint $x_0 \in X$ and if each intersection $A_\alpha \cap A_\beta$ is path-connected then the hom $\Phi : * \pi_1(A_\alpha) \longrightarrow \pi_1(X)$ is surjective. If, in addition, all triple intersections $A_\alpha \cap A_\beta \cap A_\gamma$ are also path-connected then the kernel of the map Φ equals the normal subgroup N generated by all elements of the form $i_{\alpha\beta}(\omega)\, i_{\beta\alpha}(\omega)^{-1}$ for $\omega \in \pi_1(A_\alpha \cap A_\beta)$.

Thus there is a group isomorphism $\pi_1(X) \cong \dfrac{*\,\pi_1(A_\alpha)}{N}$

__Comments :__

1) 90 % of the times one applies this to __two sets__ A_1, A_2 hence there is no extra conditions on the triple intersection.

2) Special cases:

__2a__ : if $\pi_1(A_\alpha) = 0 \ \forall \alpha$ then $\pi_1(X) = 0$ (i.e. X is also simply connected).

__2b__ : if $\pi_1(A_\alpha \cap A_\beta) = 0 \ \forall \alpha, \beta$ then $\pi_1(X) \cong *_\alpha\, \pi_1(A_\alpha)$

[in this case there are no relations]

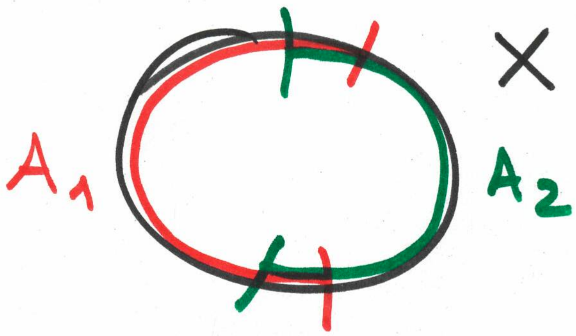
3) why the kernel ?

$$\begin{tikzcd}
 & \pi_1(A_\alpha) \arrow[dr, "j_\alpha"] & \\
\pi_1(A_\alpha \cap A_\beta) \arrow[ur, "i_{\alpha\beta}"] \arrow[d, phantom, "\cong" rotate=90] & & \pi_1(X) \\
\pi_1(A_\beta \cap A_\alpha) \arrow[r, "i_{\beta\alpha}"'] & \pi_1(A_\beta) \arrow[ur, "j_\beta"'] &
\end{tikzcd}$$

$\boxed{j_\alpha i_{\alpha\beta} = j_\beta i_{\beta\alpha}} \Longrightarrow$ relations .

4) <u>Achtung</u>! even just for the surjectivity we do need $A_\alpha \cap A_\beta$ to be path-connected.

Example: $X = S^1$ take A_1, A_2 two open arcs w/ disconnected intersection.



$\pi_1(A_1) = 0$

$\pi_1(A_2) = 0$

$\pi_1(S^1) = \mathbb{Z}$ ↯

Similar (but more sophisticated) examples show that the condition on triple intersections is also needed in general.

<u>Application 1</u>: (cf. Prop. 1.14 pg. 35)

$\pi_1(S^n) = 0$ if $n \geq 2$

Recall: S^n is the unit sphere in $\mathbb{R}^{n+1}$.

Then take: $A_1 = S^n \setminus \{\text{south pole}\}$

$A_2 = S^n \setminus \{\text{north pole}\}$

Have that A_1 and A_2 are path-connected and homeomorphic (via stereographic proj.) to $\mathbb{R}^n$ itself. $\Rightarrow \pi_1(A_1) = \pi_1(A_2) = 0$

$A_1 \cap A_2 = S^4 \setminus \{\text{north pole, south pole}\}$ is path-connected.

$\Longrightarrow \quad \pi_1(S^4) \cong 0$ $\square$
(by 2a)

<u>More generally</u>: If a top. space X is the union of two simply-connected open sets A_1, A_2 w/ $A_1 \cap A_2$ path-connected then X is simply connected.

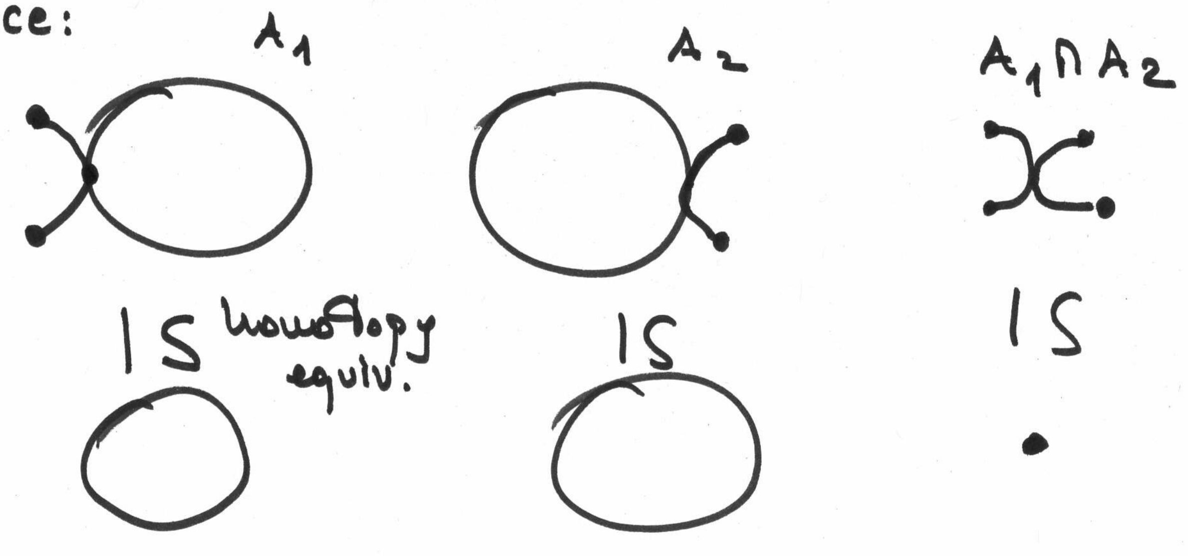
<u>Application 2</u>: consider the top. subspace of $\mathbb{R}^2$ given by $X = X' \cup X''$ where

$$X' = \{(x-1)^2 + y^2 = 1\}$$
$$X'' = \{(x+1)^2 + y^2 = 1\}$$

X is path-connected, what is $\pi_1(X)$?

$$A_1 = \{(x,y) \in X : x > -\epsilon\}$$
$$A_2 = \{(x,y) \in X : x < +\epsilon\}$$
for $0 < \epsilon < 1$.

Hence:



$\pi_1(A_1) \cong \mathbb{Z}$ $\pi_1(A_2) \cong \mathbb{Z}$ $\pi_1(A_1 \cap A_2) = 0$

Conclusion: $\pi_1(X) \cong \mathbb{Z} * \mathbb{Z}$

(2b)

More generally: if a top. space X is the union of path-connected open sets A_1, A_2 w/ $A_1 \cap A_2$ simply connected (e.g. $A_1 \cap A_2$ is contractible) then $\pi_1(X) \cong \pi_1(A_1) * \pi_1(A_2)$.