Van Kampen's Theorem - Lecture 20

heuristic (excision principle): if \( X = A \cup U B \)
and we understand (i.e., we can compute \( \pi_1 \)) of \( A, B, A \cap B \)
then we understand (i.e., we can compute \( \pi_1 \)) of \( X \)

Free Product of Groups

Let \( S \) be a set \((S \neq \emptyset)\). A word (in the
alphabet \( S \)) is a finite sequence of letters in \( S \),
possibly empty. On the set of all words \( W(S) \)
there is a natural product (juxtaposition)

\[
\begin{array}{ccc}
(C, p') & \longrightarrow & pp'
\end{array}
\]

i.e., if \( p = s_1 \ldots s_m \) \( \Rightarrow \) \( pp' = s_1 \ldots s_ms'_1 \ldots s'_n \).

Def. The free product of two groups \( H \) and \( K \)
is \( H \ast K : = W(H \sqcap K) \) where

\( p \ast p' \) if \( p' \) can be obtained from \( p \) via
a finite chain of words which are:

- \( M_1 \) if \( p = x_1 \ldots x_m \) and \( x_i, x_{i+1} \in H \) (or \( E K \))
  then I replace \( x_i x_{i+1} \) by their product \( x_i x_{i+1} \in H \) (or \( K \))
- \( M_2 \) if \( p = x_1 \ldots x_m \) and \( x_i = e_H \) (or \( x_i = e_K \))
  then I remove this letter from the string \( p \);

or the inverse of these moves:

- \( M_1' \) replace \( x \) by \( x_i x_{i+1} \) (if \( x_i x_{i+1} = x \)
  \( \in H \) or \( K \))
- \( M_2' \) insert identity element of either group.
Prop.: The product operation (as defined in $W(H \sqcup K)$) descends to the quotient $H \ast K$ and defines a group.

Proof: if $p_1 \cap p_1'$, then $p_1 p_2 \cap p_1' p_2'$ because I just apply both sets of moves!

- Identity: take the empty word
- Inverse: if $p = a_1 \ldots a_e$ set $p' = a_e^{−1} \ldots a_1^{−1}$ then $p p' = a_1 \ldots a_e a_e^{−1} \ldots a_1^{−1} \\ \gtrless a_1 \ldots a_e^{−1} e a_e^{−1} \ldots a_1^{−1} (H1) \\ \gtrless a_1 \ldots a_e^{−1} a_e^{−1} \ldots a_1^{−1} \neq \text{empty} (H2)$ except the empty word.

Prop.: Every element of $H \ast K$ can be written uniquely as a word $a_1 \ldots a_e$ where
- $a_{2i+1} \in H$ and $a_{2i} \in K \quad \forall i$
- or vice versa $a_{2i} \in H$ and $a_{2i+1} \in K \quad \forall i$
- $a_i \neq e_H, e_K \quad \forall i$.

Such form is called reduced.

Proof: exercise (induction on the length of the given word).
Cor. 1: The map \( \phi: H \rightarrow H \ast K \) defined by mapping any \( h \in H \) to the word with that one letter is an injective homomorphism (same story for \( \gamma: K \rightarrow H \ast K \)).

Proof. 1. To check that this is a homomorphism is equivalent to proving that if \( h = h_1 h_2 \) in \( H \) then \( h = h_1 h_2 \) in \( \mathbb{W}(H \ast K) \)

true by \( H(1) \) above.

2. Injective \( \phi(h) = \text{empty word} \), is reduced \( \Rightarrow h = e_H 1 u H \) \( \Box \)

Cor. 2: If \( H, K \) are both non-trivial groups then \( H \ast K \) contains an element and is not abelian.

Proof. Joke given the reduced form:

1. \( h \neq e_H 1 u H \) reduced words \( h \)
   \( K \neq e_K \) in \( K \)

2. \( h K = K h \) again by uniqueness of reduced form.
Def. We define the free group (of rank two) on $\mathbb{Z} \times \mathbb{Z}$ (i.e. the output of the construction above in the special case $H = K = \mathbb{Z}$).

\[
\begin{align*}
\text{Rmk. } & : Z = \langle a \rangle \quad \text{1st copy} \\
& \quad Z = \langle b \rangle \quad \text{2nd copy}
\end{align*}
\]

a (reduced) word $a^1 b^1 d^1 a^2 d^2 \ldots b^d e$

- Cor. 2 applies to this case: $\mathbb{Z} \times \mathbb{Z}$ is not abelian.
- $A b(\mathbb{Z} \times \mathbb{Z}) = \mathbb{Z} \oplus \mathbb{Z}$ (abelian group of rank 2).

Fact: The whole discussion above can be extended to \{ $G\alpha$ \}

\[
\begin{align*}
\alpha & \mapsto * \alpha \ G\alpha
\end{align*}
\]

In particular one can define \( \mathbb{Z} \times \ldots \times \mathbb{Z} \) (free group of rank n).

Extension of group homomorphisms: given a collection of group hom.

\[
\begin{align*}
\varphi_\alpha : & \ G\alpha \longrightarrow H \quad (\text{for some group } H) \\
\text{"there is a unique group hom that extends them"} \quad \text{i.e. } & \exists ! \quad \psi : * G\alpha \longrightarrow H
\end{align*}
\]

\[\varphi_\alpha = \psi \circ \iota_\alpha \ \forall \alpha \text{ where } \iota_\alpha : G\alpha \to G\]
Let's say you have a word $g_1 \cdots g_n$

$$
\varphi(g_1 \cdots g_n) = \varphi(g_1) \cdots \varphi(g_n)
$$

$$
= \varphi_{a_1}(g_1) \cdots \varphi_{a_n}(g_n)
$$

if $g_i \in G_{a_i}$

Note that if $g_1 \cdots g_n \sim g'_1 \cdots g'_m$ a words
then by our def. $\varphi(g_1 \cdots g_n) = \varphi(g'_1 \cdots g'_m)$
because the image, through $\varphi$, of a word
does not change when I perform one more

(by explicit def. of $\Pi_1, \Pi_2$ and axioms [prop. of group how]).

Conclusion: $\varphi$ is well-defined and (trivially)
a group homomorphism $\varphi: * G_{a} \to H$
extending the given ones.
Van Kampen: suppose \( X = \bigcup A_\alpha \) w/ \( A_\alpha \) path-connected, open in \( X \) (\( \forall \alpha \))

Suppose \( \exists x_0 \in X \) w/ \( x_0 \in \cap A_\alpha \)

basepoint for fundamental group

There exist maps induced by the inclusions:

\[ j_\alpha : \pi_1 (A_\alpha) \rightarrow \pi_1 (X) \]

\[ \xi_\alpha : \pi_1 (A_\alpha \cap A_\beta) \rightarrow \pi_1 (A_\alpha) \]

By previous discussion there is a group hom \( \Phi : \ast \pi_1 (A_\alpha) \rightarrow \pi_1 (X) \) extending the single maps \( j_\alpha \).

Theorem: If \( X \) is the union of path-connected open sets each containing the basepoint \( x_0 \in X \) and if each interset \( A_\alpha \cap A_\beta \) is path-connected then the hom \( \Phi : \ast \pi_1 (A_\alpha) \rightarrow \pi_1 (X) \) is surjective. If, in addition, all triple intersetions \( A_\alpha \cap A_\beta \cap A_\gamma \) are also path-connected then the kernel of the map \( \Phi \) equals the normal subgroup \( N \) generated by all elements of the form \( \xi_\alpha (w) \xi_\beta (w)^{-1} \) for \( w \in \pi_1 (A_\alpha \cap A_\beta) \).
Thus there is a group isomorphism \( \pi_1(\Sigma) \cong \pi_1(\Sigma_\alpha) \cong \pi_1(\Sigma_\beta) \cong \pi_1(\Sigma_\alpha \cap \Sigma_\beta) \).

Comments:

1) 90% of the times one applies this to two sets \( A_1, A_2 \) hence there is no extra condition on the triple intersection.

2) Special cases:

\[ 2a : \text{if } \pi_1(A_\alpha) = 0 \text{ for all } \alpha \text{ then } \pi_1(X) = 0 \text{ (i.e. } X \text{ is also simply connected).} \]

\[ 2b : \text{if } \pi_1(A_\alpha \cap A_\beta) = 0 \text{ for all } \alpha, \beta \text{ then } \]
\[ \pi_1(X) \cong \pi_1(A_\alpha) \cong \pi_1(A_\beta) \]

[In this case there are no relations.]

3) Why the kernel?

\[ \pi_1(A_\alpha \cap A_\beta) \]
\[ \pi_1(A_\beta \cap A_\alpha) \]
\[ \pi_1(X) \]

\[ \alpha \xrightarrow{\text{cap}} \pi_1(A_\alpha) \]
\[ \beta \xrightarrow{\text{cap}} \pi_1(A_\beta) \]

\[ \alpha \cap \beta = \beta \cap \alpha \Rightarrow \text{relations} \]
4) Ach! even just for the surjectivity we do need $A_a \cap A_B$ to be path-connected. Example: $X = S^1$ take $A_1, A_2$ two open arcs w/ disconnected intersection.

\[ \pi_1(A_1) = 0 \]
\[ \pi_1(A_2) = 0 \]
\[ \pi_1(S^1) = \mathbb{Z} \]

Similar (but more sophisticated) examples show that the condition on triple intersections is also needed in general.

**Application 1:** (cf. Prop. 1.14 pg. 35)
\[ \pi_1(S^n) = 0 \text{ if } n \geq 2 \]

Recall: $S^n$ is the unit sphere in $\mathbb{R}^{n+1}$.

Then take:
\[ A_1 = S^n \setminus \{ \text{south pole} \} \]
\[ A_2 = S^n \setminus \{ \text{north pole} \} \]

Have that $A_1$ and $A_2$ are path-connected and homotopy equivalent (via stereographic proj.) to $\mathbb{R}^n$ itself. \( \Rightarrow \pi_1(A_1) = \pi_1(A_2) = 0 \)
$A_1 \cap A_2 = S^n \setminus \{\text{north pole, south pole}\}$ is path-connected.

$\Rightarrow \pi_1(S^n) \cong 0$ (by 2a)

**Note generally**: If a top. space $X$ is the union of two simply-connected open sets $A_1, A_2$ with $A_1 \cap A_2$ path-connected then $X$ is simply-connected.

**Application 2**: Consider the top. subspace of $\mathbb{R}^2$ given by $X = X' \cup X''$ where

$X' = \{ (x-1)^2 + y^2 = 1 \}$

$X'' = \{ (x+1)^2 + y^2 = 1 \}$

$X$ is path-connected, what is $\pi_1(X)$?

$A_1 = \{ (x, y) \in X : x > -\varepsilon \}$ for $0 < \varepsilon < 1$. 

$A_2 = \{ (x, y) \in X : x < +\varepsilon \}$

Hence:

$A_1 \cong S^1$ isomoprhicequiv.

$A_2 \cong S^1$
\[ \pi_1(A_1) \cong \mathbb{Z} \quad \pi_1(A_2) \cong \mathbb{Z} \quad \pi_1(A_1 \cap A_2) = 0 \]

**Conclusion:** \[ \pi_1(\mathcal{X}) \cong \mathbb{Z} \times \mathbb{Z} \quad (26) \]

**More generally:** if a top. space \( \mathcal{X} \) is the union of path-connected open sets \( A_1, A_2 \) w/ \( A_1 \cap A_2 \) simply connected (e.g. \( A_1 \cap A_2 \) is contractible), then \( \pi_1(\mathcal{X}) \cong \pi_1(A_1) \times \pi_1(A_2) \).