Retractions and deformation retracts – Lecture 21

<u>Def.</u> Let X be a top. space, let $A \subset X$ and let $\iota: A \longrightarrow X$ be the inclusion. We say that $\rho: X \to A$ is a retraction (and A is a retract of X) if

$\rho \circ \iota = id_A$.

<u>Key fact</u>: assume that X is path-connected (which implies that A is path-connected as well, being a retract of X). Then

$$(\rho \circ \iota)_* = (id_A)_*$$

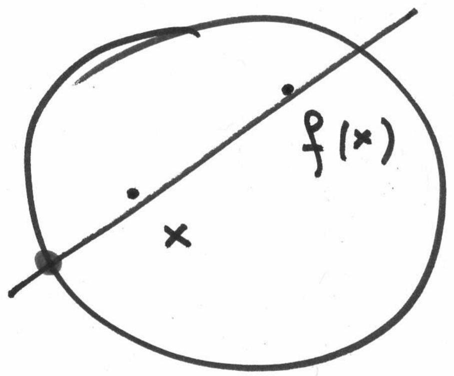
$$\rho_* \circ \iota_* = id_{\pi_1(A)}$$

$$\Longrightarrow \begin{cases} \iota_* : \pi_1(A) \longrightarrow \pi_1(X) \text{ injective} \\ \rho_* : \pi_1(X) \longrightarrow \pi_1(A) \text{ surjective} \end{cases}$$

<u>Example</u>: could $S^1 = \partial D^2$ be a retract of D^2 itself? Answer: NO, because $\pi_1(S^1) = \mathbb{Z}$, $\pi_1(D^2) = 0$ so there can be <u>no</u> injective (group hom) $\mathbb{Z} \longrightarrow 0$.

<u>Application</u>: (Brouwer fixed point theorem)
Let $D^2 \subset \mathbb{R}^2$ be the closed unit disc and let $f: D^2 \longrightarrow D^2$ continuous. Then f has a fixed point, i.e. $\exists\ x \in D^2$ w/ $f(x) = x$.

Proof. Suppose not, then $f(x) \neq x \ \forall x \in D^2$.
For any $x \in D^2$ consider the line L_x in $\mathbb{R}^2$ passing through x and $f(x)$. Let $g(x)$ be the intersection point of L_x w/ S^1 closest to x.



S^1

- $g : D^2 \to S^1$ is continuous

$$g(x) = x + \lambda \frac{x - f(x)}{|x - f(x)|}$$

where λ is selected by imposing $|g(x)| = 1$, in part.

$$\lambda = - \left\langle x, \frac{x - f(x)}{|x - f(x)|} \right\rangle + \sqrt{\left\langle x, \frac{x - f(x)}{|x - f(x)|} \right\rangle - |x|^2 + 1}$$

- $g(x) = x \quad \forall x \in S^1$

Hence g would be a retraction of D^2 onto S^1, which is impossible. $\square$

Rmk. : In part. this applies to $\mathrm{Aut}(\mathbb{D})$.

<u>Def.</u> In the setting of the previous definition, we say that A is a <u>deformation retract</u> if there is a homotopy $R : X \times I \longrightarrow X$ w/

$$\begin{cases} R(x,0) = x & \\ R(x,1) = \iota \circ \rho(x) & \forall x \in X \\ R(a,t) = a & \forall a \in A \;\; \forall t \in I \end{cases}$$

<u>Rmks</u>: 1) In part. $\iota \circ \rho : X \circlearrowleft$ is homotopic to the identity map on X.

2) saying that a space X deformation-retracts onto one of its points is strictly stronger than saying that X is contractible.

<u>Key fact</u>: assume that X is path-connected then $(\iota \circ \rho)_* = (id_X)_*$ hence

$$\iota_* \circ \rho_* = id_{\pi_1(X)}$$

So $\iota_* : \pi_1(A) \longrightarrow \pi_1(X)$

$\rho_* : \pi_1(X) \longrightarrow \pi_1(A)$ are <u>isomorphisms</u>.

<u>Example</u>: Take the torus $\underbrace{S^1_\phi \times S^1_\theta}_{X}$, let $\underbrace{S^1 \times \{\theta_0\}}_{A}$ be a meridian circle for some fixed $\theta_0 \in S^1_\theta$.

- A is a retract of X, $\rho(\phi, \theta) = (\phi, \theta_0)$
- A could not possibly be a deformation retract of X, because $\pi_1(A) \cong \mathbb{Z}$ while $\pi_1(X) \cong \mathbb{Z}^2$ so (since the two groups are <u>not</u> isomorphic) there cannot exist a map R as above.

<u>Application</u>: Wedge sums of top. spaces.

Given $\underbrace{\text{top.}}_{\text{path-connected}}$ spaces $\{X_\alpha\}$ set $X = \coprod X_\alpha / \sim$

where $\sim$ identifies $x_\alpha \in X_\alpha$ to the same point in the quotient.

$(p_1 \sim p_2$ if $p_1 = p_2$ or $p_1 = x_{\alpha_1}$ and $p_2 = x_{\alpha_2})$

Notation: $X = \bigvee_\alpha X_\alpha$.

Assume that $\forall \alpha$ the point $x_\alpha \in X_\alpha$ is a deformation retract of some open neighb. $U_\alpha \subset X_\alpha$ i.e. $\exists F_\alpha : U_\alpha \times I \to U_\alpha$

$$\begin{cases} F_\alpha(x, 0) = x \\ F_\alpha(x, 1) = x_\alpha \\ F_\alpha(x_\alpha, t) = x_\alpha \quad \forall t \in I \end{cases}$$

($\Rightarrow U_\alpha$ path-connected $\forall \alpha$)

Compute $\pi_1(X)$ using Van Kampen:

$A_\alpha := X_\alpha \vee_{\beta \neq \alpha} U_\beta$

Then: $A_{\alpha_1} \cap A_{\alpha_2} = \vee_\beta U_\beta$

$A_{\alpha_1} \cap A_{\alpha_2} \cap A_{\alpha_3} = \vee_\beta U_\beta$

but $\vee_\beta U_\beta$ is path-connected so can apply Van Kampen. Have: $\pi_1(A_{\alpha_1} \cap A_{\alpha_2}) = 0$ because $A_{\alpha_1} \cap A_{\alpha_2} = \vee_\beta U_\beta$ deformation retracts onto $[x_\alpha]$ i.e. the basepoint. Also $\pi_1(A_\alpha) \cong \pi_1(X_\alpha)$

Conclusion $\pi_1(X) \cong *_\alpha \pi_1(X_\alpha)$.

<u>Special case</u>: $\bigvee_\alpha S^1_\alpha$ (wedge of circles)

$$\pi_1\left(\bigvee_\alpha S^1_\alpha\right) \cong *_\alpha \left(\pi_1(S^1_\alpha)\right)$$

$$\cong *_\alpha \mathbb{Z}$$

Finite case: $\underbrace{S^1 \vee \dots \vee S^1}_{n \text{ times}}$ ← cf. Problem 10.8 and 10.9

$$\downarrow \pi_1$$

$$F_n = \underbrace{\mathbb{Z} * \dots * \mathbb{Z}}_{n \text{ times}}$$