Proof of Van Kampen's Theorem - Lecture 22

- Setup $X = \bigcup_\alpha A_\alpha$, w/ A_α open path-connected
  assume $\exists\, x_0 \in \bigcap_\alpha A_\alpha$ (basepoint)
  have inclusions $\varphi_\alpha : A_\alpha \longrightarrow X$
  these induce $(\varphi_\alpha)_* = j_\alpha : \pi_1(A_\alpha) \longrightarrow \pi_1(X)$
- Double intersections $A_\alpha \cap A_\beta$, also path-connected
  have inclusions $\varphi_{\alpha\beta} : A_\alpha \cap A_\beta \longrightarrow A_\alpha$
  these induce $(\varphi_{\alpha\beta})_* = \iota_{\alpha\beta} : \pi_1(A_\alpha \cap A_\beta) \longrightarrow \pi_1(A_\alpha)$
- By universal property of free products there is a unique group hom. $\Phi : * \pi_1(A_\alpha) \longrightarrow \pi_1(X)$ which extends $\{j_\alpha\}$ given above. We regard $* \pi_1(A_\alpha)$ as a set of (possibly unreduced) words W modulo an equivalence relation $\sim$.

<u>First part</u>: in the setting above the map $\Phi : * \pi_1(A_\alpha) \longrightarrow \pi_1(X)$ is <u>surjective</u>.

<u>Second part</u>: if each triple intersection $A_\alpha \cap A_\beta \cap A_\gamma$ is also path-connected, then $\operatorname{Ker}(\Phi)$ is the <u>normal</u> subgroup generated by

$$S = \{ \iota_{\alpha\beta}(\omega)\, \iota_{\beta\alpha}(\omega)^{-1} : \alpha, \beta, \omega \}$$

Rmk on part 2:

- the fact that $S \subset \ker(\Phi)$ is obvious, because if $\omega \in \pi_1(A_\alpha \cap A_\beta)$, say $\omega = [\sigma]$ then

$$\omega \in \ker \Phi \iff J_\alpha \iota_{\alpha\beta}(\omega) = J_\beta \iota_{\beta\alpha}(\omega)$$

$$\iff \varphi_\alpha(\varphi_{\alpha\beta}(\sigma)) \underset{\uparrow}{\simeq} \varphi_\beta(\varphi_{\beta\alpha}(\sigma))$$

homotopy of loops in $\Omega(X, x_0)$

but this is true because $\varphi_\alpha(\varphi_{\alpha\beta}(\sigma)) \equiv \varphi_\beta(\varphi_{\beta\alpha}(\sigma))$.

- the kernel of a group hom is always a normal sub.

so

$$S \subset \ker \Phi \implies \langle\langle S \rangle\rangle \subset \ker \Phi$$

hence for the second part it suffices to show that this inclusion is not strict, i.e. $\langle\langle S \rangle\rangle = \ker \Phi$.

Let's now proceed by proving surjectivity of Φ.

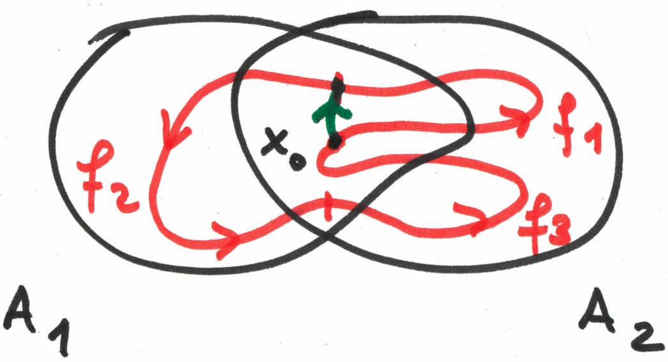
1^{st} part: surjectivity

Given $f: I \longrightarrow X$ by continuity and compactness of I we can find (using Lebesgue number) a partition
$0 = s_0 < s_1 < s_2 < \dots < s_m = 1$ such that
$f_i := f|_{[s_{i-1}, s_i]}$ $i \geq 1$ has image contained in some open set A_i. In part. note that

$$\underbrace{f_i(s_i)}_{\in A_i} = \underbrace{f_{i+1}(s_i)}_{\in A_{i+1}} \in A_i \cap A_{i+1} \qquad \left(\text{also wlog } s_j = j/m\right)$$

Note that f is equal to the (symmetric) concatenation $f_1 \cdot f_2 \cdot \dots \cdot f_m$. Then, for $i \in \{1, \dots, m-1\}$ let $g_i: I \longrightarrow A_i \cap A_{i+1}$ be a path connecting x_0 to $x_i := f_i(s_i)$. Then

$$f_1 \cdot \bar{g}_1 \in \Omega(A_1, x_0)$$
$$g_1 \cdot f_2 \cdot \bar{g}_2 \in \Omega(A_2, x_0)$$
$$\vdots$$
$$g_{m-1} \cdot f_m \in \Omega(A_m, x_0)$$



Have (using reparametrisation trick) that

$$f = f_1 \cdot \dots \cdot f_m \simeq (f_1 \cdot \bar{g}_1) \cdot (g_1 \cdot f_2 \cdot \bar{g}_2) \cdot \dots \cdot (g_{m-1} \cdot f_m)$$

but $[f_1 \cdot \bar{g}_1] \in \pi_1(X, x_0)$ also belongs to $j_1(\pi_1(A_1, x_0))$ and similarly for each factor $\Rightarrow [f] \in \Phi(\ast\, \pi_1(A_\alpha))$ $\square$

2nd part: study $\ker(\Phi)$.

Def. a factorisation of $[f] \in \pi_1(X)$ is a word (i.e. an element of W) $[f_1]\dots[f_k]$ where:

- each f_i is a loop in some A_α, based at x_0, and so $[f_i] \in \pi_1(A_\alpha)$
- f is homotopic to $f_1 \dots f_k$ as loops in X (i.e. in $\Omega(X, x_0)$).

What we proved above is the fact that any $[f] \in \pi_1(X)$ has a factorisation.

Rmk. it follows from this def. that a factorisation of $[f]$ is a word (possibly unreduced) of $*_\alpha \pi_1(A_\alpha)$ whose image through Φ is $[f]$.

Def. Two factorisations of $[f] \in \pi_1(X)$ are called equivalent if they differ by a finite sequence of these moves:

(a) moves corresponding to the equivalence relation $\sim$ in W, i.e. replacing $[f_i], [f_{i+1}] \in \pi_1(A_\alpha)$ with $[f_i \cdot f_{i+1}] \in \pi_1(A_\alpha)$, removing an identity element $[f_i] = 0 \in \pi_1(A_\alpha)$ or the inverse of these moves (cf. M1), M2), M1'), M2').)

ⓑ if $f_i \in \Omega(A_\alpha \cap A_\beta, x_0)$ replace

$$\iota_{\alpha\beta}(\omega) \equiv \underbrace{\iota_{\alpha\beta}([f_i])}_{\in \pi_1(A_\alpha)} \text{ by } \iota_{\beta\alpha}(\omega) = \underbrace{i_{\beta\alpha}([f_i])}_{\in \pi_1(A_\beta)}$$

or the inverse of this move.

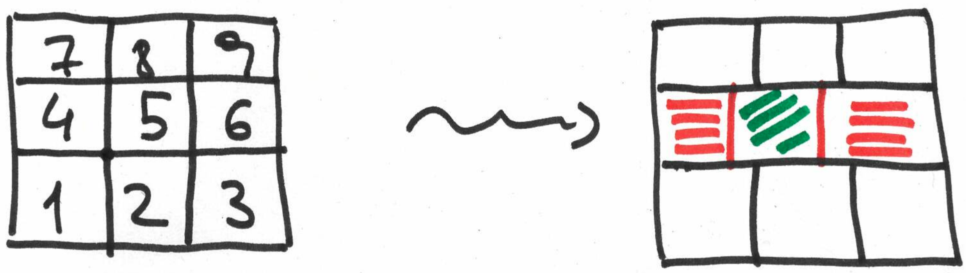
Rmk. the moves in ⓐ do not modify the equiv. class of the word in $W/\sim =: *_\alpha \pi_1(A_\alpha)$; the moves in ⓑ do modify the word in $W/\sim$ but not in $\dfrac{*_\alpha \pi_1(A_\alpha)}{N} =: Q$.

Claim ⊛ : any two factorisations of $[f] \in \pi_1(X)$ are equivalent. In part. (using the fact that any word has a unique reduced form) this claim implies the thesis ($\ker \Phi = N$, $Q \cong \pi_1(X)$) if we just take $f = c_{x_0}$ constant loop based at x_0.

Take two factorisations of $[f] \in \pi_1(X)$, $[f_1][f_2]\dots[f_k]$ and $[f_1'][f_2']\dots[f_\ell']$ and let $F: I \times I \to X$ be a homotopy of loops (based at x_0) w/

$$\begin{cases} F(s,0) = f_1 \cdots f_k(s) \\ F(s,1) = f_1' \cdots\cdots f_\ell'(s) \end{cases}$$

Take $n = d \cdot k\ell$ such that if I partition I^2 with a grid of size $1/n$ then each square $[s_{i-1}, s_i] \times [t_{j-1}, t_j] =: R_{ij}$ is mapped into some A_α, say $F(R_{ij}) = A_{ij}$. Wlog $n \geq 3$ (else refine) and perturb the vertical sides of the rectangles in the intermediate rows so that there are no more quadruple junctions in the interior.

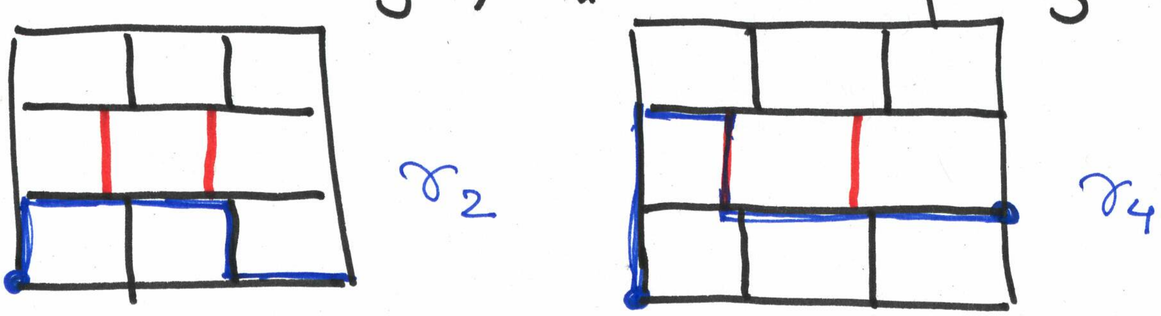


Let us relabel the new rectangles

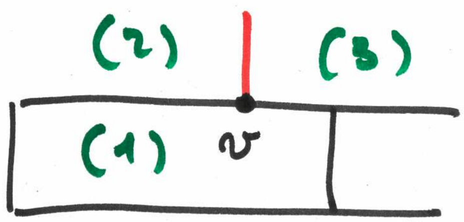
$$R_1 \quad R_2 \quad R_3 \quad \dots \quad R_{n^2}$$

starting bottom left to top right.

- Now, for $r \in \{0, 1, \dots, n^2\}$ let $\gamma_r : I \to I^2$ be a path, moving along the edges, separating $R_1 \cup R_2 \cup \dots R_r$ from the other rectangles; γ_0 is the bottom edge, γ_{n^2} is the top edge.



- Since $F|_{\{0\}\times I} \equiv x_0$ and $F|_{\{1\}\times I} \equiv x_0$ have that $\forall r$ the path in X given by $F|_{\gamma_r}$ is a loop based at x_0.
- For each vertex v of the partition, if $F(v) \neq x_0$ let $g_v : I \longrightarrow X$ be an arc connecting x_0 to $F(v)$, with $g_v(I) \subset$ the two or three A_α's such that v is a common vertex to the corresponding R_α's.



- Using $g_v, \bar{g}_v$ whenever I go through v I can factor each loop $[F|_{\gamma_r}]$ into a product of loops, based at x_0. Specifically, set $G_r : I \longrightarrow X$ be the loop, based at x_0, that is obtained from $F|_{\gamma_r}$ by introducing $g_v, \bar{g}_v$ so formally

$$G_r = F|_{\gamma_r^{(1)}} \cdot \bar{g}_{v_{(1)}} \cdot g_{v_{(1)}} \cdot F|_{\gamma_r^{(2)}} \cdot \bar{g}_{v_{(2)}} \cdot \dots$$

then there is an associated factorisation of $[F|_{\gamma_r}]$ of the form

$$[F|_{\gamma_r^{(1)}} \cdot \bar{g}_{v_{(1)}}][g_{v_{(1)}} \cdot F|_{\gamma_r^{(2)}} \cdot \bar{g}_{v_{(2)}}] \cdot \dots$$

which we write as

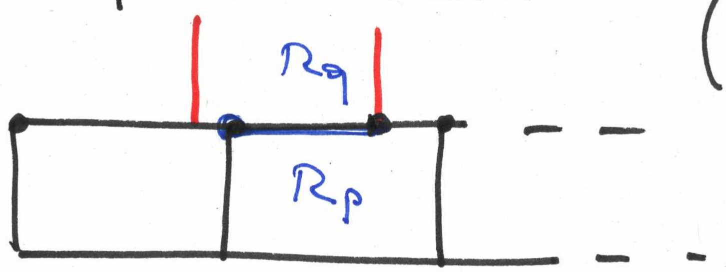
$$[G_r^{(1)}][G_r^{(2)}]\ldots[G_r^{(q_r)}]$$

Note that a priori it can be

either $[G_r^{(s)}] \in \pi_1(A_p)$

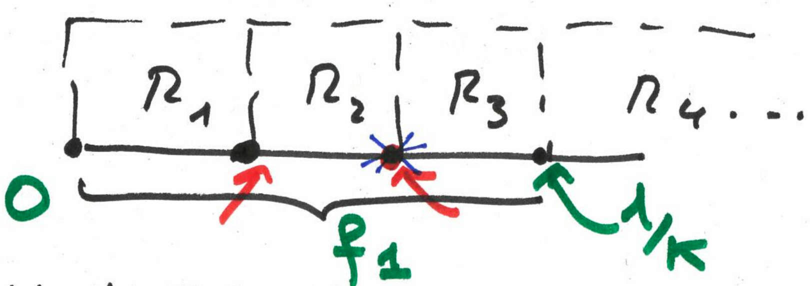
or $[G_r^{(s)}] \in \pi_1(A_q)$

if R_p, R_q are the two rectangles that have the side $\gamma_r^{(s)}$ in common.



(set a rule like first BOTTOM over TOP first LEFT over RIGHT)

Achtung: at the bottom and top sides of I^2 (i.e. when factoring $[F|_{\gamma_0}]$ and $[F|_{\gamma_{n^2}}]$) I need to perform the construction under an extra condition.



recall that $F(s,0) = f_1 \cdot \ldots \cdot f_K(s)$.

If $[f_1] \in \pi_1(A_1^*)$ and v is a vertex on the lower side such that $F(v) \neq x_0$ then wlog take $g_v, \bar{g}_v$ in $A_p \cap A_{p+1} \cap A_1^*$. Hence:

<u>Claim 1:</u> the factorisation $[f_1]\dots[f_n]$ is equivalent to the factorisation $[G_0^{(1)}]\dots[G_0^{(q_0)}]$.

<u>Pf.</u> argue as we did for surjectivity: by construction can group the factors $[G_0^{(1)}]\dots[G_0^{(q_0)}]$ into subsets corresponding to $[f_1],[f_2],\dots,[f_k]$ resp.

$$\underbrace{[G_0^{(1)}]\dots[G_0^{(d_1)}]}\,[G_0^{(d_1+1)}]\dots[G_0^{(d_2)}]\dots$$

by construction each of these factors $\in \pi_1(A_1^*)$ and $f_1 \simeq G_0^{(1)}\cdot\ldots\cdot G_0^{(d_1)}$

in $\Omega(A_1^*, x_0)$

hence the two factorisations are equivalent by simply using type ⓐ moves. $\square$

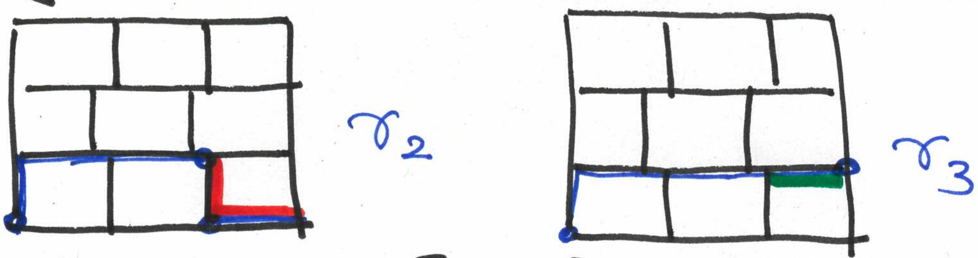
<u>Claim 2:</u> the factorisation $[f_1']\dots[f_\ell']$ is equivalent to the fact. $[G_{n^2}^{(1)}]\dots[G_{n^2}^{(q_{n^2})}]$.

<u>Proof.</u> similar to claim 1. $\square$

<u>Claim 3:</u> $\forall r\in\{0,\dots,n^2\}$ the factorisation of $[F|_{\gamma_r}]$ given above is equivalent to the fact. of $[F|_{\gamma_{r+1}}]$ given above.

Clearly, Claim 1 + Claim 2 + Claim 3 imply (by transitivity) that $[f_1] \dots [f_n]$ is equiv. to $[f_1'] \dots [f_\ell']$, hence $\circledast$ ($\Rightarrow$ $\square$).

Why is Claim 3 true? A concrete case



Let's check that $[F|\gamma_2]$ has a factorisation

$$[G_2^{(1)}] \dots [G_2^{(5)}][G_2^{(6)}]$$

that is equivalent to that of $[F|\gamma_3]$ i.e.

$$[G_3^{(1)}] \dots [G_3^{(5)}]$$

By type (b) moves can assume, wlog, that $\forall i \; \exists p$

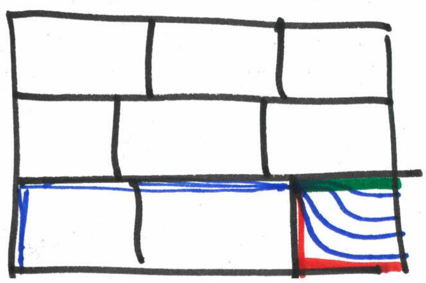
$[G_2^{(i)}], [G_3^{(i)}] \in \pi_1(A_p)$

$[G_2^{(6)}] \in$ same group as $[G_2^{(5)}]$

also by construction $G_2^{(i)} = G_3^{(i)}$ for $i = 1, 2, 3, 4$.

Conclusions come from proving $\boxed{G_2^{(5)} * G_2^{(6)} \simeq G_3^{(5)}}$ in $\pi_1(A_3)$

this follows by restriction of F (we can homotope the green part of γ_2 with the red part of γ_3).



$\square$