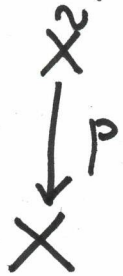


Covering Spaces and Fundamental Group - L23

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motivation: given a topological space X , can we classify all its covering spaces?



how many pairs (\tilde{X}, p) are there?

idea: we need to bring the fundamental group into play
So we'll work with path-connected, also work w/ pointed spaces i.e. pairs (X, x_0) w/ $x_0 \in X$.

Prop. 1: Let $p: (\tilde{X}, \tilde{x}_0) \rightarrow (X, x_0)$ be a cover map. Then the induced map

$$p_*: \pi_1(\tilde{X}, \tilde{x}_0) \rightarrow \pi_1(X, x_0) \text{ is injective .}$$

The subgroup $p_*(\pi_1(\tilde{X}, \tilde{x}_0))$ consists of the equiv. classes of those loops in (X, x_0) whose lifts in (\tilde{X}, \tilde{x}_0) are loops.

Proof. Let $[\tilde{f}_0] \in \ker(p_*)$: this means

$$f_0 := p(\tilde{f}_0) \stackrel{\sim}{\simeq} c_{x_0}$$

homotopy of loops in (X, x_0)

Said $F: I \times I \rightarrow X$ the homotopy in question, we know (see L19) that we can lift it to a homotopy $\tilde{F}: I \times I \rightarrow \tilde{X}$ of paths all starting at \tilde{x}_0 .

Claim:
$$\begin{cases} \tilde{F}(s, 0) = \tilde{f}_0(s) \\ \tilde{F}(s, 1) = c\tilde{z}_0 \end{cases}$$
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Now observe that $\tilde{F}(1, t) \in p^{-1}(z_0)$

hence (since fibers are discrete) $\tilde{F}(1, t) = \text{constant}$

thus $\tilde{F}(1, t) \equiv \tilde{F}(1, 1) = \tilde{z}_0$. Thus a posteriori \tilde{F} is a homotopy of loops, in part. $[\tilde{f}_0] = 0$ in $\pi_1(\tilde{X}, \tilde{z}_0)$.

Second part: if $[f] \in p_* (\pi_1(\tilde{X}, \tilde{z}_0))$, say $[f] = p_*([\tilde{g}])$. Lift f to some path \tilde{f} start at \tilde{z}_0 . Since $p\tilde{f} = f$ have that

$$[f] = p_*([\tilde{g}]) \Rightarrow f = p\tilde{f} \simeq p\tilde{g}$$

\uparrow homotopy of loops in (X, z_0)

thus by monodromy \tilde{f} is a loop $\Leftrightarrow \tilde{g}$ is a loop true!

Viceversa: if $f \in \Omega(X, z_0)$ has the property that its unique lift, starting at \tilde{z}_0 , is a loop \tilde{f}

$$\begin{aligned} \text{have } f = p\tilde{f} &\Rightarrow [f] = p_*([\tilde{f}]) \\ &\Rightarrow [f] \in p_* (\pi_1(\tilde{X}, \tilde{z}_0)). \end{aligned}$$

□

Recall that: if $p: \tilde{X} \rightarrow X$ is a covering map ^{3/6}
 then we called degree (number of sheets) of
 the covering the (locally constant) cardinality $|\bar{\pi}|$
 of any fiber $p^{-1}(x)$.

Prop. 2: The degree of a covering space
 $p: (\tilde{X}, \tilde{x}_0) \rightarrow (X, x_0)$ equals the
 index of $p_* (\pi_1(\tilde{X}, \tilde{x}_0))$ in $\pi_1(X, x_0)$.

Example a: Take $X = \tilde{X} = S^1$
 $x_0 = \tilde{x}_0 = (1, 0)$

$p(z) = z^u$ ($u \geq 1$). Then $\deg p = u$

and $p_* (\pi_1(S^1, \tilde{x}_0)) \cong n\mathbb{Z}$ \subset $\pi_1(S^1, x_0) \cong \mathbb{Z}$

Examples: recall (L12) that there is a natural
 quotient map $q: S^2 \rightarrow \mathbb{P}^2(\mathbb{R})$.

Fact: q is a covering map of degree 2. Hence

$q_* (\pi_1(S^2)) = 0$ has index 2 in $\pi_1(\mathbb{P}^2(\mathbb{R})) \cong \mathbb{Z}/2\mathbb{Z}$
 $\Rightarrow \pi_1(\mathbb{P}^2(\mathbb{R})) \cong \mathbb{Z}/2\mathbb{Z}$

Similarly, for any $n \geq 3$ have

$\pi_1(\mathbb{P}^n(\mathbb{R})) \cong \mathbb{Z}/2\mathbb{Z}$

□

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Proof.: set $H = p_* (\pi_1(\tilde{X}, \tilde{x}_0))$ and let $\mathcal{C} =$ "set of right cosets of H in G "

Claim: there is a bijection

$$\Phi: \mathcal{C} \longrightarrow p^{-1}(x_0)$$

An element of \mathcal{C} is $H[g]$ for some $[g] \in \pi_1(x, x_0) =: G$ (For instance: if $g = c_{x_0}$ then $H[g] \equiv H$ itself).

Observe that the unique lift of $h \cdot g$ starting at \tilde{x}_0 is just $h \cdot \tilde{g}$ where \tilde{g} is defined as the unique lift of g starting at \tilde{x}_0 .

h is a loop by Prop. 1

Hence the endpoint of $h \cdot \tilde{g}$, namely $h \cdot \tilde{g}(1)$ is independent of $[h]$, so set

$$\Phi(H[g]) \equiv \tilde{g}(1).$$

Surjective: given $\tilde{x}'_0 \in p^{-1}(x_0)$ let \tilde{g}' be a path in \tilde{X} connecting \tilde{x}_0 to \tilde{x}'_0 . Thus set $g' = p\tilde{g}'$ have $\Phi(H[g']) = \tilde{g}'(1) = \tilde{x}'_0$.

Injective: $\Phi(H[g_1]) = \Phi(H[g_2])$

this means $\tilde{g}_1(1) = \tilde{g}_2(1)$ hence $g_1 \equiv g_2$

lifts to a loop in \tilde{X} (namely: the loop $\tilde{g}_1 \cdot \tilde{g}_2^{-1}$).
Hence by Prop. 1 have that

$$[g_1 \cdot \bar{g}_2] \in H \equiv p_* (\pi_1(\tilde{X}, \tilde{x}_0))$$

$$\infty [g_1] \cdot [g_2]^{-1} \in H \Rightarrow H[g_1] = H[g_2]. \quad \square$$

Achtung! the subgroup $p_* (\pi_1(\tilde{X}, \tilde{x}_0))$ is not,
in general, normal in $\pi_1(X, x_0)$. This is true
under additional assumptions (stay tuned...).

Def. we say that a covering space

$p: (\tilde{X}, \tilde{x}_0) \rightarrow (X, x_0)$ is normal if

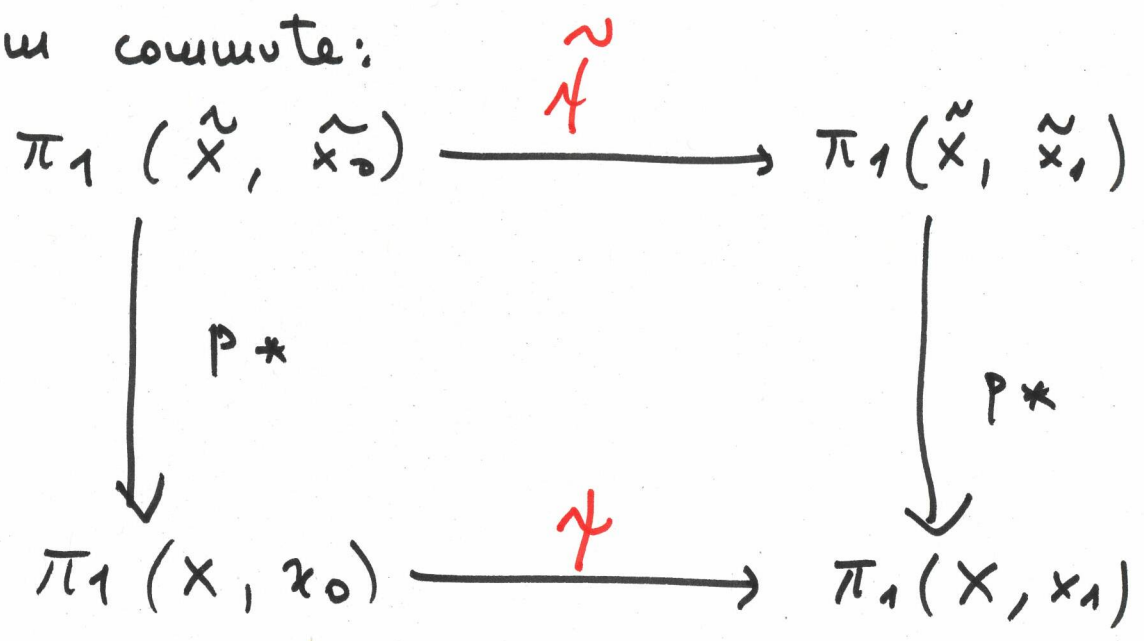
$p_* (\pi_1(\tilde{X}, \tilde{x}_0))$ is a normal subgroup in $\pi_1(X, x_0)$.

at least for some choice of $x_0 \in X, \tilde{x}_0 \in p^{-1}(x_0)$

Fact: this conclusion does not actually depend
on the choice of basepoints. why?

Prop.: Let $p: \tilde{X} \rightarrow X$ be a covering map.
 Let $\tilde{x}_0, \tilde{x}_1 \in \tilde{X}$, $x_0 = p(\tilde{x}_0)$, $x_1 = p(\tilde{x}_1)$.

There exist group isomorphisms that make this diagram commute:



PF. Just take "change of basepoint maps"
 i.e. if $\tilde{h}: I \rightarrow \tilde{X}$ $\tilde{h}(0) = \tilde{x}_1$, $\tilde{h}(1) = \tilde{x}_0$
 let $h = p \circ \tilde{h}$. It's enough to take (L16)
 $\tilde{\gamma} = \beta_{\tilde{h}}$, $\gamma = \beta_h$. □