Covering Spaces and Fundamental Group – L23

motivation: given a topological space X, can we classify all its covering spaces?

$$\begin{array}{c} \tilde{X} \\ \downarrow p \\ X \end{array}$$

how many pairs $(\tilde{X}, p)$ are there?

idea: we need to bring the fundamental group into play

So we'll work with path-connected, also work w/ pointed spaces i.e. pairs (X, x_0) w/ $x_0 \in X$.

Prop. 1: Let $p : (\tilde{X}, \tilde{x}_0) \longrightarrow (X, x_0)$ be a cover. map. Then the induced map

$$p_* : \pi_1(\tilde{X}, \tilde{x}_0) \longrightarrow \pi_1(X, x_0) \text{ is } \underline{\text{injective}}.$$

The subgroup $p_*(\pi_1(\tilde{X}, \tilde{x}_0))$ consists of the equiv. classes of those loops in (X, x_0) whose lifts in $(\tilde{X}, \tilde{x}_0)$ are loops.

Proof. Let $[\tilde{f}_0] \in \ker(p_*)$: this means

$f_0 := p(\tilde{f}_0) \simeq c_{x_0}$

homotopy of loops in (X, x_0)

Said $F : I \times I \longrightarrow X$ the homotopy in question, we know (see L19) that we can lift it to a homotopy $\tilde{F} : I \times I \longrightarrow \tilde{X}$ of paths all starting at $\tilde{x}_0$.

Claim: $\begin{cases} \tilde{F}(s,0) = \tilde{f}_0(s) \\ \tilde{F}(s,1) = c_{\tilde{x}_0} \end{cases}$

Now observe that $\tilde{F}(1,t) \in p^{-1}(x_0)$ hence (since fibers are discrete) $\tilde{F}(1,t) = \text{constant}$ thus $\tilde{F}(1,t) \equiv \tilde{F}(1,1) = \tilde{x}_0$. Thus a posteriori $\tilde{F}$ is a homotopy of loops, in part. $[\tilde{f}_0] = 0$ in $\pi_1(\tilde{X}, \tilde{x}_0)$.

Second part: If $[f] \in p_*(\pi_1(\tilde{X}, \tilde{x}_0))$, say $[f] = p_*([\tilde{g}])$. Lift f to some path $\tilde{f}$ start. at $\tilde{x}_0$. Since $p\tilde{f} = f$ have that

$$[f] = p_*([\tilde{g}]) \Rightarrow f = p\tilde{f} \underset{\uparrow}{\simeq} p\tilde{g}$$

homotopy of loops in (X, x_0)

thus by monodromy $\tilde{f}$ is a loop $\Leftrightarrow$ $\underbrace{\tilde{g} \text{ is a loop}}_{\text{true!}}$

Viceversa: If $f \in \Omega(X, x_0)$ has the property that its unique lift, starting at $\tilde{x}_0$, is a loop $\tilde{f}$ have $f = p\tilde{f} \Rightarrow [f] = p_*[\tilde{f}]$

$$\Rightarrow [f] \in p_*(\pi_1(\tilde{X}, \tilde{x}_0)).$$

□

Recall that: if $p: \tilde{X} \to X$ is a covering map then we called degree (number of sheets) of the covering the (locally constant) cardinality in $\overline{\mathbb{N}}$ of any fiber $p^{-1}(x)$.

Prop. 2: The degree of a covering space

$$p: (\tilde{X}, \tilde{x}_0) \longrightarrow (X, x_0)$$ equals the index of $p_*(\pi_1(\tilde{X}, \tilde{x}_0))$ in $\pi_1(X, x_0)$.

Example a: Take $X = \tilde{X} = S^1$

$$x_0 = \tilde{x}_0 = (1,0)$$

$p(z) = z^n$ $(n \geq 1)$. Then $\deg p = n$

and $\underbrace{p_*(\pi_1(S^1, \tilde{x}_0))}_{\equiv\, n\mathbb{Z}} < \underbrace{\pi_1(S^1, x_0)}_{\mathbb{Z}}$

Example b: recall (L12) that there is a natural quotient map $q: S^2 \longrightarrow \mathbb{P}^2(\mathbb{R})$.

Fact: q is a covering map of degree 2. Hence $\underbrace{q_*(\pi_1(S^2))}_{=0}$ has index 2 in $\pi_1(\mathbb{P}^2(\mathbb{R}))$

$$\Longrightarrow \pi_1(\mathbb{P}^2(\mathbb{R})) \cong \mathbb{Z}/2\mathbb{Z}$$

Similarly, for any $n \geq 3$ have

$$\pi_1(\mathbb{P}^n(\mathbb{R})) \cong \mathbb{Z}/2\mathbb{Z}$$

$\square$

__Proof.__: set $H = p_*(\pi_1(\tilde{X}, \tilde{x}_0))$ and let $\mathcal{C}$ = "set of right cosets of H in G"

Claim: there is a bijection

$$\Phi : \mathcal{C} \longrightarrow p^{-1}(x_0)$$

An element of $\mathcal{C}$ is $H[g]$ for some $[g] \in \pi_1(X, x_0)) =: G$ (For instance: if $g = c_{x_0}$ then $H[g] \equiv H$ itself).

Observe that the unique lift of $h \cdot g$ starting at $\tilde{x}_0$ is just $\tilde{h} \cdot \tilde{g}$ where $\tilde{g}$ is defined as the unique lift of g starting at $\tilde{x}_0$.

$\tilde{h}$ is a loop by Prop. 1

Hence the endpoint of $\tilde{h} \cdot \tilde{g}$, namely $\tilde{h} \cdot \tilde{g}(1)$ is independent of $[h]$, so set

$$\boxed{\Phi(H[g]) \equiv \tilde{g}(1).}$$

__Surjective:__ given $\tilde{x}_0' \in p^{-1}(x_0)$ let $\tilde{g}'$ be a path in $\tilde{X}$ connecting $\tilde{x}_0$ to $\tilde{x}_0'$. Thus set $g' = p\tilde{g}'$ have $\Phi(H[g']) = \tilde{g}'(1) = \tilde{x}_0$.

__Injective:__ $\Phi(H[g_1]) = \Phi(H[g_2])$ this means $\tilde{g}_1(1) = \tilde{g}_2(1)$ hence $g_1 \cdot \overline{g_2}$

lifts to a loop in $\tilde{X}$ (namely: the loop $\tilde{g}_1 \cdot \overline{\tilde{g}_2}$).
Hence by Prop. 1 have that

$$[g_1 \cdot \overline{g}_2] \in H \equiv p_* (\pi_1(\tilde{X}, \tilde{x}_0))$$

so $[g_1] \cdot [g_2]^{-1} \in H \Rightarrow H[g_1] = H[g_2]$. $\square$

<u>Achtung</u>! the subgroup $p_* (\pi_1(\tilde{X}, \tilde{x}_0))$ is <u>not</u>, in general, normal in $\pi_1(X, x_0)$. This is true under additional assumptions (stay tuned...).

<u>Def.</u> we say that a covering space $p: (\tilde{X}, \tilde{x}_0) \longrightarrow (X, x_0)$ is <u>normal</u> if $p_* (\pi_1(\tilde{X}, \tilde{x}_0))$ is a normal subgroup in $\pi_1(X, x_0)$, at least for some choice of $x_0 \in X$, $\tilde{x}_0 \in p^{-1}(x_0)$.

<u>Fact</u>: this conclusion does <u>not</u> actually depend on the choice of basepoints. why?

<u>Prop.</u>: Let $p : \tilde{X} \longrightarrow X$ be a covering map. Let $\tilde{x}_0, \tilde{x}_1 \in \tilde{X}$, $x_0 = p(\tilde{x}_0)$, $x_1 = p(\tilde{x}_1)$. There exist group isomorphisms that make this diagram commute:

$$\begin{tikzcd}
\pi_1(\tilde{X}, \tilde{x}_0) \arrow[r, "\tilde{\psi}"] \arrow[d, "p_*"] & \pi_1(\tilde{X}, \tilde{x}_1) \arrow[d, "p_*"] \\
\pi_1(X, x_0) \arrow[r, "\psi"] & \pi_1(X, x_1)
\end{tikzcd}$$

<u>Pf.</u> Just take "change of basepoint maps" i.e. if $\tilde{h} : I \longrightarrow \tilde{X}$ $\tilde{h}(0) = \tilde{x}_1$, $\tilde{h}(1) = \tilde{x}_0$ let $h = p\tilde{h}$. It's enough to take (L16) $\tilde{\psi} = \beta_{\tilde{h}}$, $\psi = \beta_h$. $\square$