Lifting criterion and existence of coverings – L24

Prop. Let $p : (\tilde{X}, \tilde{x}_0) \longrightarrow (X, x_0)$ be a covering space and let $f : (Y, y_0) \longrightarrow (X, x_0)$ be a continuous map; assume the domain Y is connected and locally path-connected. Then: a lift $\tilde{f} : (Y, y_0) \longrightarrow (\tilde{X}, \tilde{x}_0)$ exists if and only if $f_*(\pi_1(Y, y_0)) \subset p_*(\pi_1(\tilde{X}, \tilde{x}_0))$.

Rmks: (a)

$$\begin{tikzcd} & (\tilde{X}, \tilde{x}_0) \arrow[d, "p"] \\ (Y, y_0) \arrow[ur, dashed, "\tilde{f}"] \arrow[r, "f"] & (X, x_0) \end{tikzcd}$$

from one implication for free: if there exists a lift $\tilde{f}$ then we can write

$$f = p\tilde{f}$$

$$\Longrightarrow \quad f_* = p_*(\tilde{f}_*)$$

$$\Longrightarrow \quad \text{image } f_* \subset \text{image } p_*$$

So we only need to construct $\tilde{f}$ given the inclusion.

(b) recall that (as always) $X, \tilde{X}$ are assumed to be path-connected. Concerning Y note that

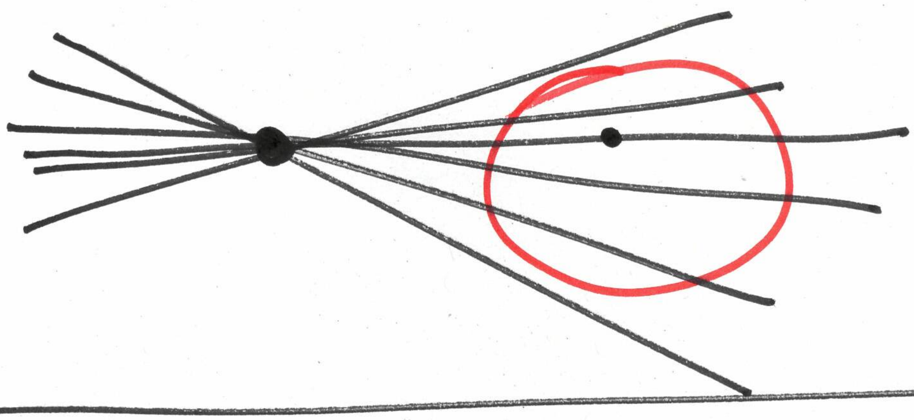
$$\text{path-connected} \not\Longrightarrow \text{locally path-connected}$$

In general, that's why we need both assumptions.

For instance, take $X \subset \mathbb{R}^2$ the subspace

$$X = \{ (x,y) \in \mathbb{R}^2 : ax + by = 0 \quad a \in \mathbb{Z},\ b \in \mathbb{Z} \}$$

This ↑ is path-connected (a union of lines through the origin) but NOT locally path-connected e.g. $(1,0) \in X$ has arbitrarily small neighborhoods that are not path-connected: if $\epsilon < 1$ then $B_\epsilon(1,0) \cap X$ is disconnected.



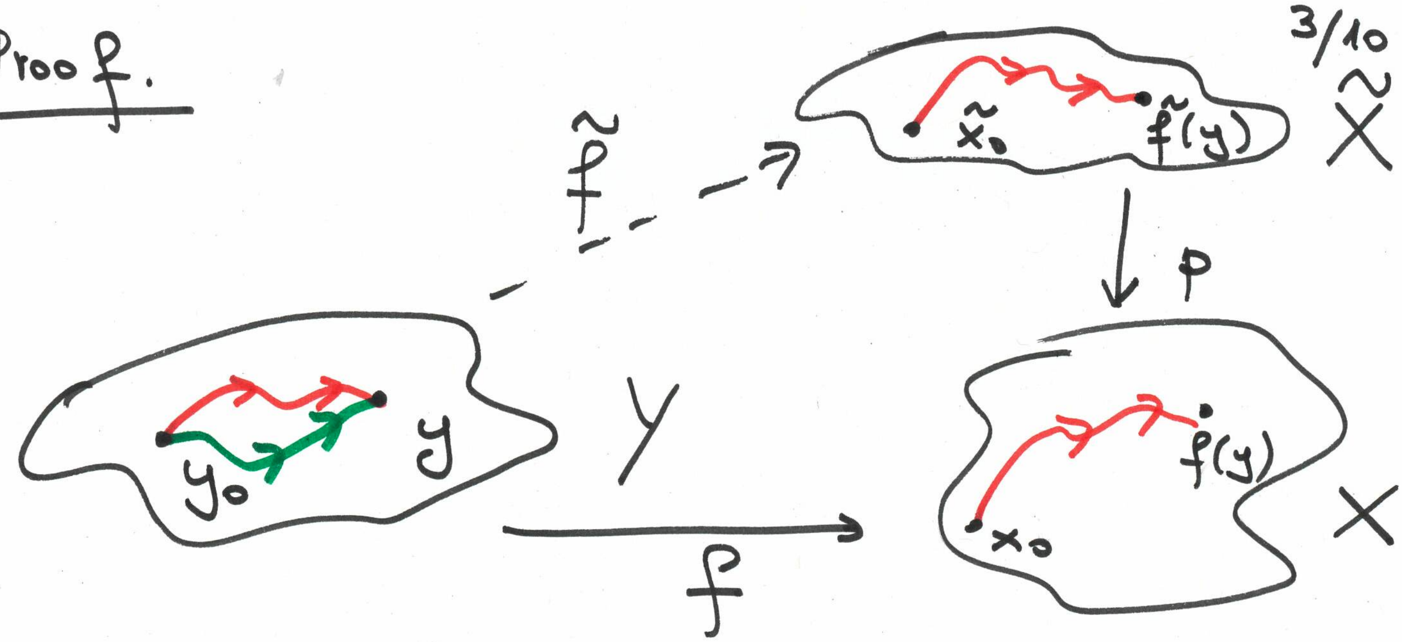
Ⓒ recovering what we already know:

if $Y = I$ then it's always true that

$$f_*(\pi_1(Y, y_0)) \subset p_*(\pi_1(X, x_0))$$

so thm. applies and ensures existence of a lift on the path $f : I \to X$ starting at a given height $\tilde{x}_0$.

Proof.



how to define $\tilde{f}(y)$? Let $\gamma : I \to Y$ join $\gamma(0) = y_0$ to $\gamma(1) = y$. Consider $f\gamma : I \to X$ and lift this path to $\widetilde{f\gamma} : I \to \tilde{X}$ starting at $\tilde{x}_0$. Define $\boxed{\tilde{f}(y) := \widetilde{f\gamma}(1).}$

Check 1: the definition does not depend on γ.

Take another connecting path γ'. Consider

$$f\gamma' \cdot \overline{f\gamma} = \underbrace{f(\gamma' \cdot \overline{\gamma})}_{h_0}$$

is a loop in (X, x_0)

so since image$(f_*) \subset$ image(p_*) there have $[h_0] \in p_*(\pi_1(\tilde{X}, \tilde{x}_0))$ i.e.

$h_0 \simeq h_1$ where $h_1 = p\tilde{h}_1$ ← loop in $(\tilde{X}, \tilde{x}_0)$

↑ homotopy of loops in (X, x_0)

Take this homotopy, say F, and lift it to a homotopy of paths all starting at $\tilde{x}_0$. Usual argument (using that fibers are discrete) gives $\tilde{F}(1,t) \equiv \tilde{F}(1,1) = \tilde{h}_1(1) = \tilde{x}_0$. So $\tilde{F}$ is actually a homotopy of loops, in particular if we define $\widetilde{\overline{f\gamma}}$ the lift of $\overline{f\gamma}$ starting at $\widetilde{f\gamma'}(1)$ we get

$$\tilde{F}(s,0) \underset{\substack{\uparrow \\ \text{uniqueness of} \\ \text{lifts}}}{=} \widetilde{f\gamma'} \cdot \widetilde{\overline{f\gamma}} \text{ is a loop, hence}$$

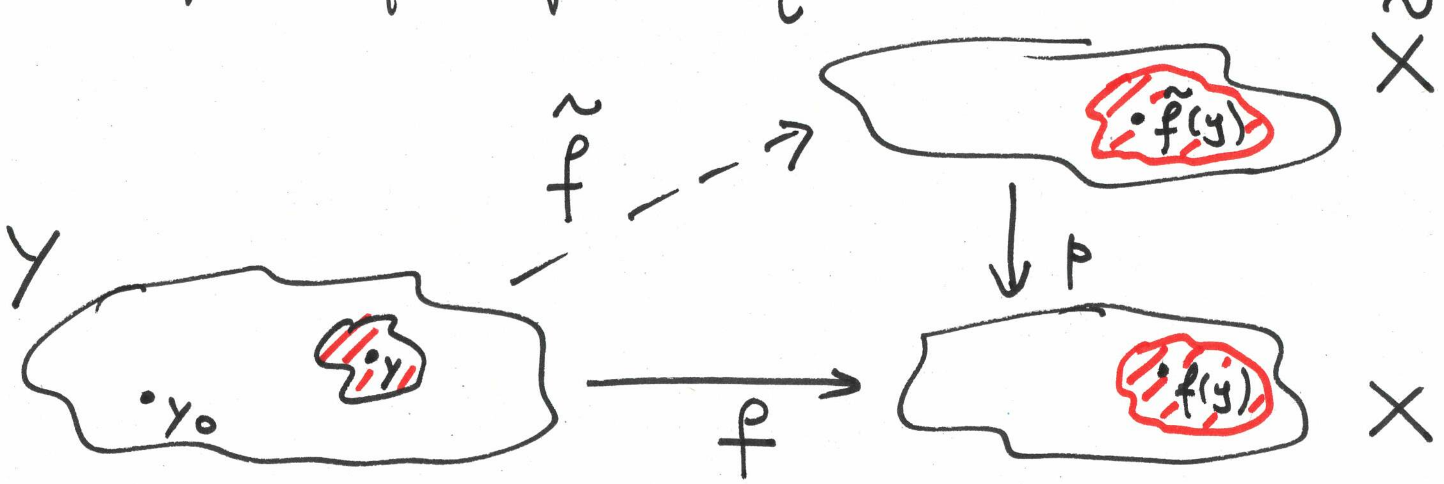
$$\widetilde{\overline{f\gamma}}(1) = \tilde{x}_0 \quad \text{so}$$

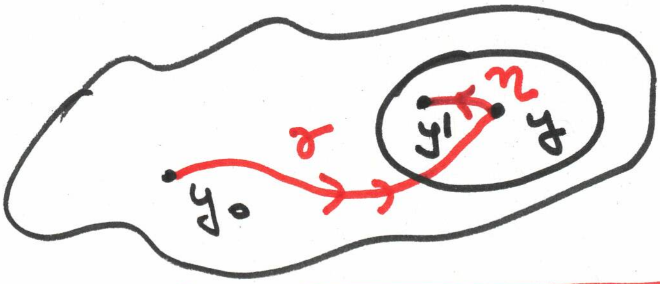
(again by uniqueness of lifts) $\widetilde{\overline{f\gamma}} = \overline{\widetilde{f\gamma}}$, thus

$$\widetilde{f\gamma}(1) = \widetilde{f\gamma'}(1) \Longrightarrow \tilde{f} \text{ is well-defined.}$$

<u>Check 2</u>: $\tilde{f}$ is continuous.

Given $y \in Y$, take $U \subset X$ an open neighbourhood of $f(y) \in X$ that is evenly covered; in part. $\exists\, \tilde{U} \subset \tilde{X}$ open neighb. of $\tilde{f}(y)$ s.t. $p|_{\tilde{U}} : \tilde{U} \to U$ is a homeomorphism. Take V a open neighb. of $y \in Y$ w/ $f(V) \subset U$. Using the fact that V can be chosen path-c. given any $y' \in V$ join y_0 to y' passing through y i.e. take paths of the form $\gamma \cdot \eta$.





$f(\gamma\cdot\eta)$ lifts to
$\widetilde{f\gamma}\cdot\widetilde{f\eta}$
where $\widetilde{f\eta} = p^{-1} f\eta$

$$\Rightarrow \quad \boxed{\tilde{f}|_V = p|_{\tilde{U}}^{-1} \circ f} \qquad \square$$

Recall (from L18) the unique lifting property:

<u>Prop.</u> Given a covering space $p : \tilde{X} \to X$ and a map $f : Y \to X$ if two lifts $\tilde{f}_1, \tilde{f}_2 : Y \to \tilde{X}$ agree at one point $y \in Y$ and Y is connected then $\tilde{f}_1 \equiv \tilde{f}_2$.

(<u>proof.</u> observe that $Z = \{ y \in Y : \tilde{f}_1(y) = \tilde{f}_2(y) \}$ is open and closed in Y)

Classification of covering spaces

idea : Galois-type correspondence between

$$\left\{ \begin{array}{c} \text{(connected) covering} \\ \text{spaces of } X \end{array} \right\} \qquad \left\{ \begin{array}{c} \text{subgroups} \\ \text{of } \pi_1(X) \end{array} \right\}$$

Two key questions:

- Is it true that for any subgroup $H < \pi_1(X)$ there is a (connected) covering space $\tilde{X}$ of X w/ $p_*(\pi_1(\tilde{X})) = H$?

  (surjectivity of the correspondence above)

- are two covering spaces inducing the same subgroup of $\pi_1(X)$ necessarily "isomorphic" in a suitable sense?

  (injectivity of the correspondence above)

Answer to both questions is YES under reasonable technical assumptions.

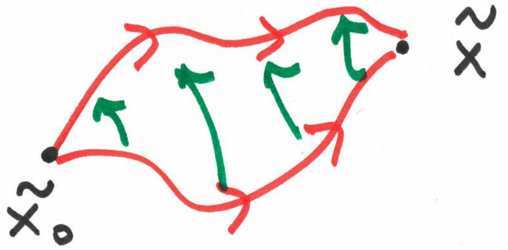
<u>Prop.</u> Suppose X is <u>path-connected</u>
<u>locally path-connected</u>
and <u>semilocally simply-connected</u>

Then for every subgroup $H < \pi_1(X, x_0)$ there is a covering space $p: X_H \longrightarrow X$ such that $p_*(\pi_1(X_H, \tilde{x}_0)) = H$ for suitable $\tilde{x}_0 \in X_H$.

<u>Rmks.</u>: Semilocally simply ~~path~~-connected means: every point $x \in X$ has a (path-connected) neighb. such that the inclusion-induced map $\pi_1(U, x) \longrightarrow \pi_1(X, x)$ is trivial ($\equiv$ the ZERO map!).

- Clearly seen to be necessary for the existence of a simply-connected cover.
- implied by the condition that X is locally simply-connected (i.e. $\forall x \in X$ $\exists$ arb. small $U_i = U_i(x)$ that are simply-connected).

<u>Proof.</u> Case when H = trivial group.
If $p: (\tilde{X}, \tilde{x}_0) \longrightarrow (X, x_0)$ is a simply-connected covering space then (Prop. 1.6, Exercise 8.4) $\forall \tilde{x} \in \tilde{X}$ there is a unique homotopy class of paths from $\tilde{x}_0$ to $\tilde{x}$.



We take this fact to suggest the definition of $\tilde{X}$.

Given (X, x_0) define:

- $\tilde{X} = \{ [\gamma] : \gamma : I \longrightarrow X \text{ w/ } \gamma(0) = x_0 \}$

$[\gamma]$ denotes a homotopy of paths w/ both endpoints fixed $(*)$

- $p : \tilde{X} \longrightarrow X$ the map $p([\gamma]) = \gamma(1)$

by $(*)$ if $[\gamma_1] = [\gamma_2]$ then in part. $\gamma_1(1) = \gamma_2(1)$ so p is well-defined.

Also note that $p : \tilde{X} \to X$ is <u>surjective</u>

- define a topology on the set $\tilde{X}$ as follows:

Let $\mathcal{U}$ be the collection of all (path-connected) open sets $U \subset X$ such that $\pi_1(U) \to \pi_1(X)$ is trivial. Note that $\mathcal{U}$ is "stable under inclusion" i.e. if $U \in \mathcal{U}$ and $V \subset U$ open and path-connected then

$$\pi_1(V) \longrightarrow \pi_1(U) \longrightarrow \pi_1(X)$$

is also trivial. Hence, under our assumptions

$\mathcal{U}$ is a <u>basis</u> for the topology on X.

Given $\gamma : I \longrightarrow X$ w/ $\gamma(0) = x_0$ and $U \in \mathcal{U}$, $U \ni \gamma(1)$ we define" a neighbourhood of $[\gamma]$ in $\tilde{X}$ by declaring

$$U_{[\gamma]} = \left\{ [\gamma \cdot \eta] : \eta \text{ is a path in } U \text{ w/ } \gamma(1) = \eta(0) \right\}$$

Now can check (using criterion in L3) $U_{[\gamma]}$ is a basis for a topology on $\tilde{X}$, and from there $p : \tilde{X} \longrightarrow X$ is a covering map. (I omit the details here, see pg. 64)

- Check that <u>$\tilde{X}$ is simply-connected</u>.

$\circledast$ $\tilde{X}$ is path-connected: we show that any point $[\gamma] \in \tilde{X}$ can be joined to $[c_{x_0}] \in \tilde{X}$.

How? take $\gamma_t : I \longrightarrow X$ the <u>truncation</u> of γ at time t i.e.

$$\gamma_t(s) = \begin{cases} \gamma(s) & 0 \le s \le t \\ \gamma(t) & t \le s \le 1 \end{cases}$$

Note $\gamma_1 = \gamma$ while $\gamma_0 = c_{x_0}$

hence $t \longmapsto [\gamma_t]$ is a C^0- path connecting $[c_{x_0}]$ to $[\gamma]$.

$\circledast\circledast$: $\pi_1(\tilde{X}, [c_{x_0}]) = 0$

by injectivity of p_* (seen in L 23) it is enough to check that $p_*(\pi_1(\tilde{X}, [c_{x_0}]) = 0$. So let $[\gamma] \in p_*(\pi_1(\tilde{X}, [c_{x_0}]))$. By def. γ is a loop in (X, x_0) that lifts to a loop in $(\tilde{X}, [c_{x_0}])$. But the <u>unique</u> lift of $t \longmapsto \gamma(t)$ starting at $[c_{x_0}]$ is just $t \longmapsto [\gamma_t]$. Hence $[\gamma_1] = [c_{x_0}]$ which means that $\gamma_1 = \gamma$ is homotopic (as loops) to the constant loop c_{x_0} in X. Namely: we have proven that $[\gamma] = 0$ thus $(\tilde{X}, [c_{x_0}])$ has trivial fundamental group. $\square$