Existence of Coverings – L25

Prop. Suppose X is path-connected
locally path-connected
semilocally simply-connected

For every subgroup $H < \pi_1(X, x_0)$ there is a covering space $p: X_H \longrightarrow X$ such that $p_*(\pi_1(X_H, \tilde{x}_0)) = H$ for a suitably chosen basepoint $\tilde{x}_0$.

Recall: last time we proved it when $H = 0$, now we exploit that construction to handle the case of general H.

Proof. Let $\tilde{X}$ be the (universal) cover we constructed in L24 i.e.

$$\tilde{X} = \{ [\gamma] : \gamma : I \longrightarrow X \ / \ \gamma(0) = x_0 \}$$

put an equivalence relation on $\tilde{X}$ by declaring

$$[\gamma] \sim [\gamma'] \quad \text{iff} \quad \begin{cases} \gamma(1) = \gamma'(1) \\ [\gamma \cdot \bar{\gamma}'] \in H \end{cases}$$

$[c_{x_0}]$ ↙

$0 \in H \implies \sim$ reflexive

$h \in H \Rightarrow h^{-1} \in H \implies \sim$ symmetric

$h_1, h_2 \in H \Rightarrow h_1 h_2 \in H \implies \sim$ transitive

Then set $X_H := \tilde{X}/_\sim$ namely the quotient space of $\tilde{X}$ w.r.t. this relation. Related map:

$$\begin{array}{ccc} \tilde{X} & & \\ {\scriptstyle q}\downarrow & \searrow^{p} & \\ X_H = \tilde{X}/_{\sim} & \overset{p_H}{\dashrightarrow} & X \end{array}$$

recall $p([\gamma]) = \gamma(1)$

p_H is just gotten by 'passing to the quotient' namely

$p_H(q[\gamma]) = \gamma(1)$

This is well-defined because

$$[\gamma_1] \sim [\gamma_2] \implies \gamma_1(1) = \gamma_2(1).$$

The map $p_H : X_H \longrightarrow X$ is actually continuous by "universal property of quotients" due to the fact that $p = p_H \circ q$ and we already know (last time ⊕ Exercise 12.5) that $p : \tilde{X} \to X$ is itself continuous. Also p_H is surjective because p is.

Check 1: $p_H : X_H \longrightarrow X$ is a covering map.

We already know that for any fixed point $x \in X$ if $p^{-1}(x) = \{[\gamma_j]\}$ $j \in J$ then

$\exists$ U evenly covered neighb. of $x \in X$

$\exists$ $U_{[\gamma_j]}$ open neighb. of $[\gamma_j] \in \tilde{X}$ w/ the property $p|_{U[\gamma_j]} : U_{[\gamma_j]} \longrightarrow U$ homeo.

Now, there are two cases:

i) if $[\gamma_{j_1}] \sim [\gamma_{j_2}]$ then $\forall \eta : I \to U$

w/ $\eta(0) = \underbrace{\gamma_{j_1}(1) = \gamma_{j_2}(1)}_{\equiv x}$

have $[\gamma_{j_1} \cdot \eta] \sim [\gamma_{j_2} \cdot \eta]$

(just because $\overline{\gamma_{j_2} \cdot \eta} = \bar{\eta} \cdot \bar{\gamma}_{j_2}$ and in the

product $(\gamma_{j_1} \cdot \eta) \cdot \overline{(\gamma_{j_2} \cdot \eta)} = \gamma_{j_1} \cdot \underbrace{\eta \cdot \bar{\eta}} \cdot \bar{\gamma}_{j_2}$

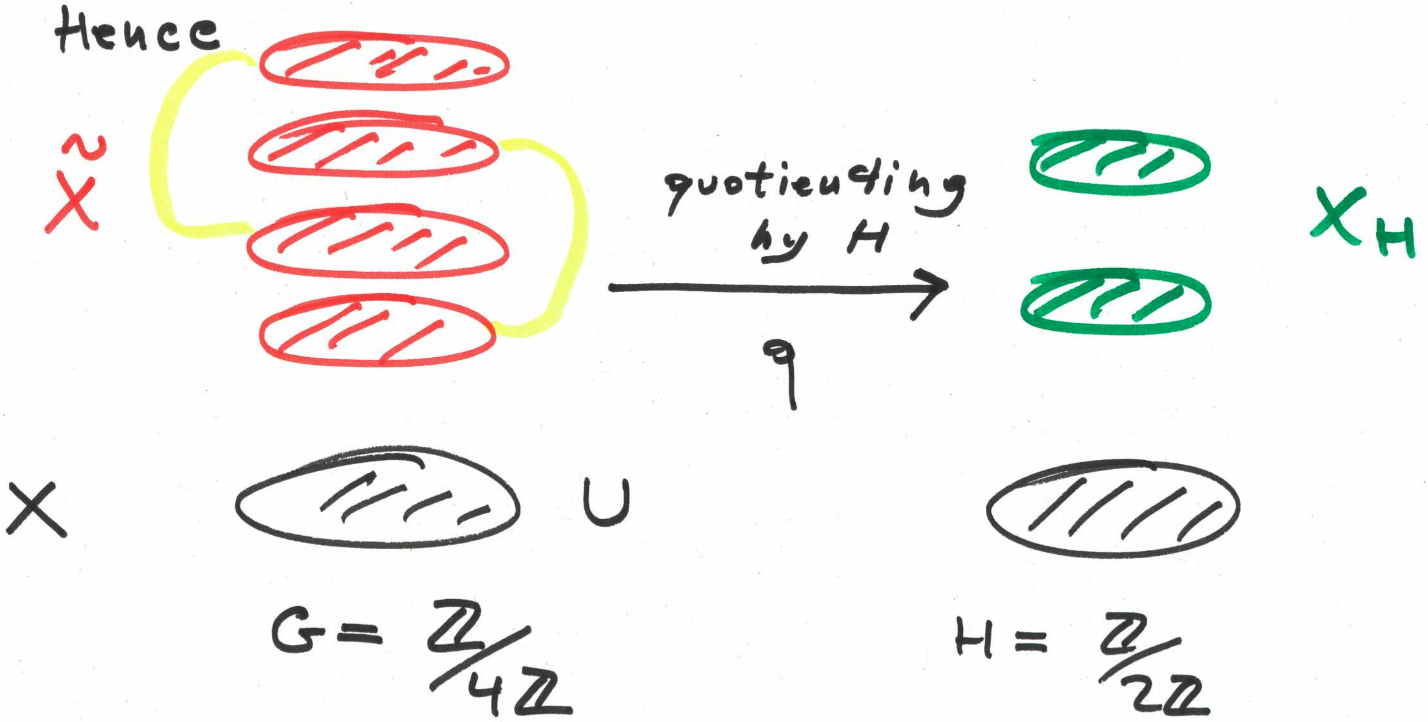
cancels out modulo homotopy

ii) if $[\gamma_{j_1}] \not\sim [\gamma_{j_2}]$ then $\forall \eta : I \to U$

w/ $\eta(0) = \gamma_{j_1}(1) = \gamma_{j_2}(1)$

have $[\gamma_{j_1} \cdot \eta] \not\sim [\gamma_{j_2} \cdot \eta]$.

Hence



We have shown (through the analysis above) that $q : \tilde{X} \longrightarrow X_H$ is a covering map, in part.

$$\forall j \in J \quad q|_{U_{[\gamma_j]}} : U_{[\gamma_j]} \longrightarrow \underbrace{q(U_{[\gamma_j]})}_{V_j}$$

is a homeomorphism; hence the same set $U \subset X$ is evenly covered w.r.t. $p_H : X_H \longrightarrow X$ and $p_H|_{V_j} : V_j \longrightarrow U$ is a homeomorphism because it satisfies

$$p_H|_{V_j} = p|_{U_{[\gamma_j]}} \circ \left(q|_{U_{[\gamma_j]}} \right)^{-1} .$$

<u>Check 2</u>: set $\tilde{x}_0 \in X_H$ the equivalence class (w.r.t. $\sim$) of the constant path $[c_{x_0}]$.

Noted that X_H is path-connected, we claim that

$$\boxed{(p_H)_* (\pi_1 (X_H, \tilde{x}_0)) = H .}$$

Let γ be a loop in X based at x_0, let

$\tilde{\gamma} : I \longrightarrow \tilde{X}$ be $\underbrace{\tilde{\gamma}(t) := [\gamma_t]}_{\text{unique lift of } \gamma \text{ in } \tilde{X} \text{ starting at } [c_{x_0}]}$

$\gamma_H : I \longrightarrow X_H$ be $\gamma_H(t) := q([\gamma_t])$

$$\equiv \underbrace{q(\tilde{\gamma}(t))}_{\text{unique lift of } \gamma \text{ starting at } \tilde{x}_0 \text{ i.e. at } q([c_{x_0}]) \text{ in } \tilde{X}/\sim = X_H}$$

$[\gamma] \in H \iff [\gamma] \sim [c_{x_0}]$

$$\iff q([\gamma]) = q([c_{x_0}])$$

$$\begin{matrix} q([\gamma]) & = & q([c_{x_0}]) \\ \Vert\Vert & & \Vert\Vert \\ q([\gamma_1]) & & q([\gamma_0]) \\ \Vert & & \Vert \\ \gamma_H(1) & & \gamma_H(0) \end{matrix}$$

hence $[\gamma] \in H \iff \gamma_H(1) = \gamma_H(0)$

Now, by Prop. 1 in L23 have that $[\gamma] \in (p_H)_*(\pi_1(X_H, \tilde{x}_0))$ if and only if the unique lift of γ starting at $\tilde{x}_0$ is itself a loop in $(X_H, \tilde{x}_0)$

$\iff \gamma_H : I \longrightarrow X_H$ is a loop

$\iff [\gamma] \in H$. Conclusion as claimed:

$$\boxed{(p_H)_*(\pi_1(X_H, \tilde{x}_0)) = H.}$$

$\square$