Uniqueness of covering spaces – L26

(→ 2nd part of Galois correspondence)

Def. Given a top. space X and covering spaces
$p_1 : \tilde{X}_1 \to X$
$p_2 : \tilde{X}_2 \to X$
we say that the first is isomorphic to the second if there is a homeomorphism $f : \tilde{X}_1 \to \tilde{X}_2$ such that $p_1 = p_2 f$.
(f: isomorphism of covering spaces).

Commutative diagram:

$$\begin{tikzcd} \tilde{X}_1 \arrow[rr, "f"] \arrow[dr, "p_1"'] & & \tilde{X}_2 \arrow[dl, "p_2"] \\ & X & \end{tikzcd}$$

Fact: this defines an equivalence rel. in the class of covering spaces of a given top. space X, i.e.

- $(p_1, \tilde{X}_1) \sim (p_1, \tilde{X}_1)$ $\quad f = \mathrm{id}$
- $(p_1, \tilde{X}_1) \underset{f}{\sim} (p_2, \tilde{X}_2) \Longrightarrow (p_2, \tilde{X}_2) \underset{f^{-1}}{\sim} (p_1, \tilde{X}_1)$
- $(p_1, \tilde{X}_1) \underset{f}{\sim} (p_2, \tilde{X}_2)$, $(p_2, \tilde{X}_2) \underset{g}{\sim} (p_3, \tilde{X}_3) \Longrightarrow (p_1, \tilde{X}_1) \underset{g \cdot f}{\sim} (p_3, \tilde{X}_3)$

<u>Prop.</u> (uniqueness) If X is connected and locally path-connected, then two path-connected covering spaces $p_1: \tilde{X}_1 \to X$, $p_2: \tilde{X}_2 \to X$ are <u>isomorphic</u> via an isomorphism $f: \tilde{X}_1 \to \tilde{X}_2$ taking a basepoint $\tilde{x}_1 \in p^{-1}(x_0)$ to a basepoint $\tilde{x}_2 \in p^{-1}(x_0)$ if and only if

$$(*) \quad \boxed{(p_1)_*(\pi_1(\tilde{X}_1, \tilde{x}_1)) = (p_2)_*(\pi_1(\tilde{X}_2, \tilde{x}_2))}.$$

<u>Comment</u>: In part. if we work w/ pointed spaces (X, x_0) and

$$(\tilde{X}_1, \tilde{x}_1) \text{ w/ } p_1: \tilde{X}_1 \to X \quad p_1(\tilde{x}_1) = x_0$$
$$(\tilde{X}_2, \tilde{x}_2) \text{ w/ } p_2: \tilde{X}_2 \to X \quad p_2(\tilde{x}_2) = x_0$$

and consider isomorphisms of pointed covering spaces then to any subgroup $H < \pi_1(X, x_0)$ there is only <u>one</u> <u>associated covering</u> space, up to isomorphism.

Proof. $\boxed{\Rightarrow}$ if there is an isomorphism $f: \tilde{X}_1 \to \tilde{X}_2$ as above, have
$$\begin{cases} p_1 = p_2 f \\ p_2 = p_1 f^{-1} \end{cases}$$

by functoriality
$$\begin{cases} (p_1)_* = (p_2)_* f_* \\ (p_2)_* = (p_1)_* (f_*)^{-1} \end{cases}$$

these two give
$$\begin{cases} (p_1)_* (\pi_1(\tilde{X}_1, \tilde{x}_1)) \subset (p_2)_* (\pi_1(\tilde{X}_2, \tilde{x}_2)) \\ (p_2)_* (\pi_1(\tilde{X}_2, \tilde{x}_2)) \subset (p_1)_* (\pi_1(\tilde{X}_1, \tilde{x}_1)) \end{cases}$$

$$\Rightarrow (p_1)_* (\pi_1(\tilde{X}_1, \tilde{x}_1)) = (p_2)_* (\pi_1(\tilde{X}_2, \tilde{x}_2)).$$

$\boxed{\Leftarrow}$ Suppose $\circledast$ holds.

$$\begin{tikzcd} & (\tilde{X}_2, \tilde{x}_2) \arrow[d, "p_2"] \\ (\tilde{X}_1, \tilde{x}_1) \arrow[ur, dashed, "\tilde{p}_1"] \arrow[r, "p_1"'] & (X, x_0) \end{tikzcd}$$

Lifting criterion: $\exists!\ \tilde{p}_1 : (\tilde{X}_1, \tilde{x}_1) \longrightarrow (\tilde{X}_2, \tilde{x}_2)$ that lifts p_1 i.e. $\boxed{p_1 = p_2 \tilde{p}_1.}$

Sym

$$\begin{tikzcd} & (\tilde{X}_1, \tilde{x}_1) \arrow[d, "p_1"] \\ (\tilde{X}_2, \tilde{x}_2) \arrow[ur, dashed, "\tilde{p}_2"] \arrow[r, "p_2"'] & (X, x_0) \end{tikzcd}$$

$\exists!\ \tilde{p}_2 : (\tilde{X}_2, \tilde{x}_2) \longrightarrow (\tilde{X}_1, \tilde{x}_1)$ w/ $\boxed{p_2 = p_1 \tilde{p}_2}$

By boxes $\Longrightarrow$ $\begin{cases} p_1 = p_2 \tilde{p}_1 \\ p_2 = p_1 \tilde{p}_2 \end{cases}$

$$\boxed{\begin{aligned} p_2 &= p_2 \tilde{p}_1 \tilde{p}_2 \\ p_2 &= p_2 \, \mathrm{id} \end{aligned}}$$

$$\begin{tikzcd}
 & (\tilde{X}_2, \tilde{x}_2) \arrow[d, "p_2"] \\
(\tilde{X}_2, \tilde{x}_2) \arrow[ur, dashed, "\mathrm{id}", "\tilde{p}_1 \tilde{p}_2"'] \arrow[r, "p_2"'] & (X, x_0)
\end{tikzcd}$$

Both maps in red also send $\tilde{x}_2$ to $\tilde{x}_2$, hence (uniqueness of lifts) get $\tilde{p}_1 \tilde{p}_2 = \mathrm{id}|_{\tilde{X}_2}$. Similarly $\tilde{p}_2 \tilde{p}_1 = \mathrm{id}_{\tilde{X}_1}$. Thus $\tilde{p}_1, \tilde{p}_2$ are both homeomorphisms, hence isomorphisms of covering spaces. $\square$

Thm. (Galois correspondence) Let X be path-connected, locally path-connected, semilocally simply-connected.

Then there is a bijection between the set of basepoint-preserving isomorphism classes of path-connected pointed covering spaces $p: (\tilde{X}, \tilde{x}_0) \to (X, x_0)$ and the set of subgroups of $\pi_1(X, x_0)$, obtained by associating the subgroup $p_*(\pi_1(\tilde{X}, \tilde{x}_0))$ to the covering space in question.

Universal Cover

Def. A simply-connected covering space of a top. space X is called universal cover.

Facts:

if X is path-connected, locally path-connected, semi locally simply-connected $\Rightarrow$ u.c. exists

if X is path-connected, locally path-connected $\Rightarrow$ u.c. (when it exists) is unique up to isomorphism

under these assumptions we call it THE universal cover

Universal why? the universal cover is actually a cover of any other covering space of the same X in the following sense:

$$\begin{tikzcd} & \tilde{X}' \arrow[d, "p'"] \\ \tilde{X} \arrow[ur, dashed, "q"] \arrow[r, "p"'] & X \end{tikzcd}$$

check: $q: \tilde{X} \to \tilde{X}'$ is itself a covering map

Deck Transformations

Def. Let $p: \tilde{X} \to X$ be a covering space. We call deck transformation (of that covering space) an isomorphism of the covering itself, i.e. a homeomorphism $g: \tilde{X} \circlearrowleft$ w/ $\boxed{pg = p.}$

Two basic facts:

1. the set of deck transformations (of a given covering space) forms a group under composition denoted $G(\tilde{X})$.

2. assume $X, \tilde{X}$ are path-connected and locally path-connected. Then a deck transformation has no fixed points (unless it is the identity).

$$\begin{tikzcd} & \tilde{X} \arrow[d, "p"] \\ \tilde{X} \arrow[ur, dashed, "g"] \arrow[r, "p"'] & X \end{tikzcd}$$

- A deck transformation g is in part. a lift of p ($pg = p$).
- The identity is also a lift

If g has a fixed point, say $g(\tilde{x}) = \tilde{x}$ then $g = id$ at that point $\Longrightarrow$ $g \equiv id$

(uniqueness of lifts)

Differently phrased: a deck transformation is uniquely determined by its value at a point.

Examples:

a) $X = S^1 \quad \tilde{X} = \mathbb{R} \quad p: \tilde{X} \longrightarrow X$

$$t \longmapsto (\cos(2\pi t), \sin(2\pi t))$$

$$G(\tilde{X}) = \text{integer translations} \cong \mathbb{Z}.$$

b) $X = S^1 \quad \tilde{X} = S^1 \quad p: \tilde{X} \longrightarrow X$

$$z \longmapsto z^n$$

$$G(\tilde{X}) = \text{rotations of an integer multiple of } 2\pi/n$$

$$\cong \mathbb{Z}/n\mathbb{Z}$$

Achtung! in both cases above

$$\frac{\pi_1(X)}{p_*(\pi_1(\tilde{X}))} \cong G(\tilde{X})$$

Thm. Let $p: (\tilde{X}, \tilde{x}_0) \longrightarrow (X, x_0)$ be a path-connected covering space of the path-connected, locally path-connected space X and let H be the subgroup $p_*(\pi_1(\tilde{X}, \tilde{x}_0)) < \pi_1(X, x_0)$.

Then:

(i) this covering space is geometrically normal i.e.

$\forall x \in X$, $\forall \tilde{x}, \tilde{x}' \in p^{-1}(x)$ there is a deck transformation $f : \tilde{X} \circlearrowleft$ w/ $f(\tilde{x}) = \tilde{x}'$

if and only if it is algebraically normal

in the sense that $H \triangleleft G$.

(ii) If the covering is normal then $G(\tilde{X})$ is isomorphic to $\pi_1(X, x_0)/H$.

Comments:

- if $\tilde{X}$ is a universal cover then

$$\underbrace{\pi_1(X, x_0)}_{\text{fundamental group of the base}} \cong \underbrace{G(\tilde{X})}_{\text{automorphisms of a universal cover}}$$

- in general if $p : \tilde{X} \to X$ is not normal then the replacement for (ii) is

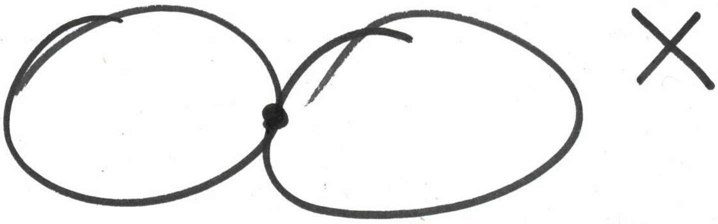
(ii)': $G(\tilde{X})$ is isomorphic to the quotient

$$N(H)/H$$ where $N(H)$ is the normaliser of H inside $\pi_1(X, x_0)$

√ • $N(H)$ is the largest (intermediate) subgroup $K < G$ where H is normal

• $N(H) = G \iff H \triangleleft G$

An example of a covering space that is <u>not</u> normal.



$$\pi_1(X) \cong \underbrace{\mathbb{Z} * \mathbb{Z}}_{G}$$

$H = \langle ab \rangle$

$(\cong \mathbb{Z})$

- H not normal in G: $\quad a H a^{-1} \not\subset H$

$$a(ab)a^{-1}$$
$$= \underbrace{a^2 b a^{-1}}_{\text{reduced word not in } H}$$

- invoke 'abstract existence result' for X_H (cf. picture pg. 58 Hatcher's book).