Fundamental Theorem on deck Transformations – L27

Thm: Let X be path-connected, locally path-connected, and let $p: (\tilde{X}, \tilde{x}_0) \longrightarrow (X, x_0)$ be a path-connected covering space. Set $G = \pi_1(X, x_0)$, $H = p_*(\pi_1(\tilde{X}, \tilde{x}_0))$.

(i) $H \triangleleft G \iff \forall\ \tilde{x}_0, \tilde{x}_1 \in p^{-1}(x_0)$

$$\exists\ \tau \in G(\tilde{X}) \text{ w/ } \tau(\tilde{x}_0) = \tilde{x}_1 .$$

(ii) In case i) have $$\frac{\pi_1(X, x_0)}{p_*(\pi_1(\tilde{X}, \tilde{x}_0))} \cong G(\tilde{X}) .$$

Comment: based on final Prop. in L23

the covering is geom. normal

$(\forall x \in X, \forall\ \tilde{x}, \tilde{x}' \in p^{-1}(x_0)$ $\exists\ \tau \in G(\tilde{X})$ w/ $G(\tilde{x}) = \tilde{x}'$ $) \iff \forall\ \tilde{x}_0, \tilde{x}_1 \in p^{-1}(x_0)$ $\exists\ \tau \in G(\tilde{X})$ w/ $\tau(\tilde{x}_0) = \tilde{x}_1$.

Lemma (geometric effects of conjugacy)

In the setting above, if $g = [\gamma]$ for $\gamma = p \circ \tilde{\gamma}$ where $\tilde{\gamma}: I \longrightarrow \tilde{X}$ connects $\tilde{x}_0 \in p^{-1}(x_0)$ to $\tilde{x}_1 \in p^{-1}(x_0)$ have

$$g^{-1} H g = p_*(\pi_1(\tilde{X}, \tilde{x}_1)) .$$

Proof (lemma)

For notational convenience set $H_0 = p_*(\pi_1(\tilde{X}, \tilde{x}_0))$

$H_1 = p_*(\pi_1(\tilde{X}, \tilde{x}_1))$

First note that $\boxed{g^{-1} H_0 g \subset H_1}$ obvious

the point here is: why equality?

by sym $g H_1 g^{-1} \subset H_0$

$\times g^{-1}$ $\boxed{H_1 \subset g^{-1} H_0 g}$

$\Rightarrow g^{-1} H_0 g = H_1$ $\square$

Proof (thm).

(i) $H \lhd G \iff \forall g \in G \;\; g^{-1} H g = H$

$\underset{\text{(lemma)}}{\iff} \forall \tilde{x}_0, \tilde{x}_1 \in p^{-1}(x_0)$ have that

$\boxed{p_*(\pi_1(\tilde{X}, \tilde{x}_0)) = p_*(\pi_1(\tilde{X}, \tilde{x}_1))}$

$\iff \forall \tilde{x}_0, \tilde{x}_1 \in p^{-1}(x_0)$

$\exists \tau \in G(\tilde{X})$ w/

$\tau(\tilde{x}_0) = \tilde{x}_1$

$\boxed{\Leftarrow}$ a deck transformation is (in part.) a homeomorphism, and homeomorphisms induce isomorphisms at the level of π_1

$\boxed{\Rightarrow}$ Lifting Criterion

$$\begin{tikzcd} & (\tilde{X}, \tilde{x}_1) \arrow[d, "p"] \\ (\tilde{X}, \tilde{x}_0) \arrow[ur, dashed, "\tau"] \arrow[r, "p"'] & (X, x_0) \end{tikzcd}$$

τ is a continuous map that makes diagram commute, also $\tau(\tilde{x}_0) = \tilde{x}_1$. Finally, τ is a homeomorphism because we can similarly build its inverse. $\square$

(ii) Consider the map

$$\varphi: \pi_1(X, x_0) \longrightarrow G(\tilde{X})$$
$$[\gamma] \longmapsto \tau_{[\gamma]}$$

how is $\tau_{[\gamma]}$ built? Take γ, lift it to $\tilde{\gamma}$ in $\tilde{X}$ starting at $\tilde{x}_0$, set $\tilde{x}_1 := \tilde{\gamma}(1)$.

Consider diagram as above

$$\begin{tikzcd} & (\tilde{X}, \tilde{x}_1) \arrow[d, "p"] \\ (\tilde{X}, \tilde{x}_0) \arrow[ur, dashed, "\tau_{[\gamma]}"] \arrow[r, "p"'] & (X, x_0) \end{tikzcd}$$

- φ is a group homomorphism

$g = [\gamma] \rightsquigarrow \tilde{\gamma} \rightsquigarrow$ endpoint $\tilde{x}_1$ $\quad \tau_{[\gamma]}$

$g' = [\gamma'] \rightsquigarrow \tilde{\gamma}' \rightsquigarrow$ endpoint $\tilde{x}_1'$ $\quad \tau_{[\gamma']}$

$\varphi(gg')$ is obtained by lifting $\gamma \cdot \gamma'$ to a path in $\tilde{X}$ starting at $\tilde{x}_0$. Such (unique) lift is $\tilde{\gamma} \cdot \tau_{[\gamma]}(\tilde{\gamma}')$. The endpoint is $\boxed{\tau_{[\gamma]}(\tilde{x}_1')}$
so $\varphi(gg')$ is the (unique) deck transformation associated to this point, i.e. $\tau_{[\gamma]}(\tilde{x}_1')$, hence it must be equal to $\underbrace{\tau \circ \tau'}$ because
this
is a deck transformation, that makes the same diagram commute

$$\begin{tikzcd}
 & (\tilde{X}, \boxed{}) \arrow[d, "p"] \\
(\tilde{X}, \tilde{x}_0) \arrow[ur, dashed] \arrow[r, "p"'] & (X, x_0)
\end{tikzcd}$$

Conclusion: $\quad \varphi(gg') = \varphi(g) \circ \varphi(g')$.

- φ is <u>surjective</u>

say $\tau \in G(\tilde{X})$, look at $\tau(\tilde{x}_0) =: \tilde{x}_1$

take $\tilde{\gamma} : I \longrightarrow \tilde{X}$ w/ $\begin{cases} \tilde{\gamma}(0) = \tilde{x}_0 \\ \tilde{\gamma}(1) = \tilde{x}_1 \end{cases}$

$\gamma = p(\tilde{\gamma})$ apply construction above to define the deck transformation $\tau_{[\gamma]}$. Now

$$\begin{tikzcd} & (\tilde{X}, \tilde{x}_1) \arrow[d, "p"] \\ (\tilde{X}, \tilde{x}_0) \arrow[ur, "\tau_{[\gamma]}"', "\tau"] \arrow[r, "p"'] & (X, x_0) \end{tikzcd}$$

by uniqueness of lifts $\tau = \tau_{[\gamma]} = \varphi(g)$.

- $\operatorname{Ker} \varphi = H$

$\tau_{[\gamma]} \in \operatorname{Ker} \varphi \iff \tau_{[\gamma]} = id$ in $G(\tilde{X})$

$\iff [\gamma] \in \pi_1(X, x_0)$ lifts to a loop in $\tilde{X}$

construction above

$\iff [\gamma] \in H \equiv \pi_1(\tilde{X}, \tilde{x}_0)$

<u>Conclusion</u>:

$$\begin{tikzcd} \pi_1(X, x_0) \arrow[r, "\varphi"] \arrow[dr] & G(\tilde{X}) \\ & \dfrac{\pi_1(X, x_0)}{p_*(\pi_1(\tilde{X}, \tilde{x}_0))} \arrow[u, dashed, "\bar{\varphi}"'] \end{tikzcd}$$

$\square$